

MATHEMATICS

FOR
MIDDLE SCHOOLS

BOOK III • PART I



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For Middle Schools

BOOK III PART I
A Textbook for Class VIII

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Foreword

This book is Part I of Book III in the series 'Mathematics for Middle Schools'. Books I and II prepared by the NCERT in 1977 and 1978, respectively, have been extremely well received by the users as evidenced from the feed back received by us. The philosophy, approach and style of this book is also the same as adopted in Books I and II.

The first draft of this book was prepared by Prof. I. B. S. Passi and Dr. L. R. Vermani of the Kurukshetra University. The draft was then reviewed by a group of teachers and subject specialists in a workshop held at the NIE Campus. The final editing was done by Prof. S. D. Chopra, retired Senior Professor and Head, Department of Mathematics, Kurukshetra University. In this task, he received most valuable support from Shri R. S. Kothari and Dr. Atma Ram Sahu of the Department of Education in Science and Mathematics (DESM) of the Council. Mention should be made of Dr. S. K. S. Gautam who assisted the editorial team in more ways than one. The answers were also provided by him. The content-editing of the Hindi version was done by Shri Mahendra Shanker. Special mention should be made of Dr. R. P. Gupta, DESM, who made a most valuable contribution throughout in the preparation of the book. His valuable suggestions have played a vital role in bringing out this book in the present form. I am grateful to each one of them for their dedication and diligence in bringing out this book in a short time.

The ultimate judge of the worth of any book is, of course, the community of its users, namely, the students and the teachers. The NCERT will welcome their observations with gratitude so as to incorporate further possible improvements in the next edition of the book.

SHIB K. MITRA
Director

National Council of Educational
Research and Training

New Delhi
May 1979

Preface

It gives me great pleasure to place Part I of Book III, *Mathematics for Middle Schools*, before the teachers and students of Class VIII. When part II of this book is published shortly, it will complete this series of mathematics books for the middle classes. Books I and II for Classes VI and VII respectively were edited very ably by Prof. Manmohan Singh Arora. Judging from the comments received from the users, these books have been received very well. When Prof. Arora left for Bahrain early in January this year, on a Unesco assignment, the responsibility of editing Part I of Book III was given to me. It was hoped that on account of my association with the production of this series of books from the very beginning, I would be able to maintain the style and standard set by Prof. Arora. I have been assisted in my task by Shri R. S. Kothari and Dr. A. R. Sahu of the Department of Education in Science and Mathematics (DESM), NCERT. How well we have succeeded in our task can be judged only by the users.

The first draft of the manuscript was prepared by Dr. I. B. S. Passi, Professor, and Dr. L. R. Vermani, Reader, Department of Mathematics, Kurukshetra University. I had the opportunity to discuss with Dr. Passi some parts of his draft quite frequently while it was being written. This draft was then shown to practising teachers and experts at a workshop held for this purpose from January 9 to 12, 1979 at the NCERT Campus. The editors also participated in the workshop and had the benefit of discussions with the participants. The editors have tried to incorporate as many of the suggestions of the participants as possible. I am sure the book has improved greatly by these suggestions.

The present volume consists of seven units : I Real Numbers, II Exponents and Radicals, III Algebraic Expressions, IV Special Products and Factors, V Linear Equations and Inequations, VI Use of Tables, and VII Sets. Units I and VII deserve special mention. Unit I introduces irrational numbers as non-recurring, non-terminating decimals. It is

hoped that enough explanation has been given to make the concept intelligible to thirteen-year-olds. Some appeal to a limiting process, however disguised, is essential for a proper understanding of the concept. Unit VII introduces the notion of Sets. The set language is now basic for the clear expression of many concepts in mathematics and its applications. Therefore, this notion has been illustrated by a large number of examples.

Part II of Book III will deal with the topics of Geometry, Commercial Arithmetic and Statistics. These topics are expected to be taught in the later half of the school year.

The salient features of this book are much the same as those of Books I and II. However, some remarks on the general objective kept in view in these books are in order. In the fifties of this century, revolutionary changes took place in the West in mathematics curricula for the schools, and many countries switched on to the so-called 'Modern Mathematics'. These changes in the curricula were brought to the notice of the teachers and administrators in India through an extensive programme of summer institutes through the sixties. Under the influence of this programme, the curricula and books for school mathematics were re-written, at the national level and in many states. Modern mathematics did not mean any new mathematics. It meant a shift of emphasis from the intuitive to the axiomatic method, from mathematical content and the development of mathematical skills to the learning of the mathematical structure of the various operations and more precise new terminology. However, experience with the new mathematics courses did not prove to be a happy one. These courses created difficulties for a vast majority of the teachers, the students and the parents. Learning of mathematics in the schools got bogged down in the language of axiomatics and the new terminology, and the learning of mathematical skills suffered. All this led to another extensive revision of school syllabi in mathematics in the recent years, and the present series of textbooks is a consequence of this.

The present series of textbooks strikes a healthy balance between the so-called modern and traditional mathematics. Unnecessary emphasis on the use of the language of axiomatics and new terminology is avoided. These are brought to the notice of the reader and left at that. The subject matter is treated more or less intuitively. Further, it has been kept in view that the mathematics which we teach a child should be easily assimilable

by him and have as close a relation to his environment as possible. It should also be relevant to the needs of our society keeping in view the goals of our development.

Some advice on the learning of mathematics is given to the students in the prefaces to Books I and II. It is suggested that the students read this advice again.

The present book contains nearly a hundred carefully selected solved illustrative examples. There are over four hundred exercises to give the child enough practice in applying the concepts and methods learnt. These exercises are graded except in the sets of miscellaneous exercises. Harder sections in the text and harder exercises are marked with an asterisk.

The editors are grateful to Dr. S. K. Singh Gautam, DESM, NCERT for providing a large number of additional exercises and helping with the answers to all the exercises.

I am very grateful to the members of the mathematics wing of the Department of Education in Science and Mathematics for their continued courtesy and assistance throughout my association with the present series of textbooks. I am thankful to the Assistant Editors without whose ungrudging hard work it would have been impossible to finish the editing work in so short a time. My special thanks are due also to Dr. R. P. Gupta, DESM, NCERT who has not only provided all the administrative support necessary, but also read the manuscript at various stages of the work and made valuable suggestions.

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List of Symbols

$+$	:	addition
$-$	:	subtraction
$\times$	:	multiplication
$\div$	:	division
$=$	:	is equal to
$<$	:	is less than
$>$	:	is greater than
$\leq$	:	is less than or equal to
$\geq$	:	is greater than or equal to
$\sqrt{\quad}$	:	square root of
$\sqrt[n]{\quad}$	:	nth root of
$\%$	:	per cent
$\in$	:	belongs to
$\notin$	:	does not belong to
---	:	such that
$\emptyset$	:	null set
$\subset$	:	is a subset of (or is contained in)
$\supset$	:	is a superset of (or contains)
$\nsubseteq$	:	not a subset of
$\cup$	:	union
$\cap$	:	intersection

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11. Express the decimal 0.625 in the form $\frac{p}{q}$ and express your answer in lowest terms.
12. Give five examples of non-terminating recurring decimals.
13. Simplify

$$\left\{ \left(\frac{6}{8} + \frac{9}{11} \right) - 0.35 \right\} \div \frac{3}{5}$$

14. Find three rational numbers between 0.1 and 0.2.
15. Which of the following integers are odd ?
158, -1715, 26170, 987003
16. Which of the following integers are even ?
201, 202, 203, 204, 205
17. If n is an even integer, is the successor of n odd or even ?
18. If n is an odd integer, is the successor of n odd or even ?
19. By squaring the following integers, verify that their squares are even :
10, -2402, 236, -576
20. By squaring the following integers, verify that the square of an odd integer is odd :
-21, 23, -753, 801
21. Are the following statements true or false ?
 - (i) Sum of squares of two even integers is even.
 - (ii) Sum of squares of two odd integers is odd.

1.2 Introduction

We have learnt about natural numbers, integers and rational numbers in earlier classes. Natural numbers can be used for three main purposes in daily life. The first is in *counting*, when we say that there are thirty boys in a class or the population of a village is 673, and so on. The second is in *measurement of magnitudes*, when we talk of 1 kilogram of sugar or 2 litres of milk or 30 quintals of wheat, and so on. The third is in *labelling* when we assign roll numbers to students in a class, or assign roll numbers to candidates appearing in an examination, etc.

We found it necessary to introduce the negative integers to be able to carry out the operation of subtraction in natural numbers always.

The necessity for introducing rational numbers arises when we wish to divide a certain number of things equally into a given number of parts, for example, when we wish to divide 3 oranges equally among 5 boys. The necessity for rational numbers arises also in many other problems in daily life, for example, in measuring a piece of cloth, or the distance between two points, or the length of a room, which is not a whole multiple of our unit of measurement. We now see that the rational numbers do not suffice for this purpose.

We have learnt in Class VII how to represent rational numbers by points on the number line. Figure 1.1 shows the number line OAX with some rational numbers marked on it.

We recall that when we say that the points A, B, C represent the rational numbers 1, 2, $\frac{7}{4}$, we mean that

the lengths OA, OB, OC are $1, 2, \frac{7}{4}$ units respectively. In other words, in representing rational numbers on a line, we construct line segments on the line whose measures in terms

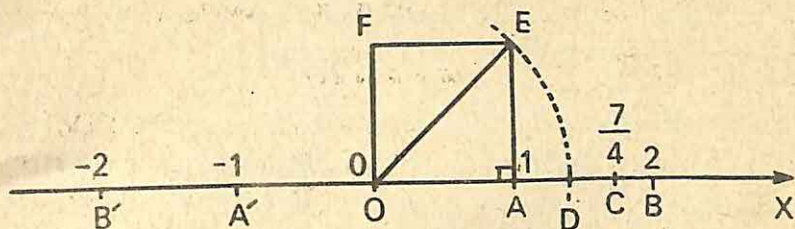


Fig. 1.1 : The number line

of a unit length are these rational numbers. We now ask the question : Can the measure of *every* line segment on the line be expressed as a rational number ? The answer to this question is 'No.' Let us consider the following example.

Let us construct the square OAEF on OA as one side. (See Fig. 1.1) Join O to E. Then $OA = AE = 1$ unit. What is the length of the diagonal OE ? We note that OAE is a right triangle with the right angle at A. By Pythagoras theorem*

$$OE^2 = OA^2 + AE^2 = 1^2 + 1^2 = 2$$

Thus, OE is a length whose square is equal to 2. In other words, the length of OE is the square root of 2. Let us agree to denote the square root of 2 by the symbol $\sqrt{2}$.

With O as centre and OE as radius, draw an arc cutting OX at D. Then $OD = OE = \sqrt{2}$. Then OD is a line segment in the number line whose length is $\sqrt{2}$. We learn in the next

*Pythagoras Theorem : In any right triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides. You will learn this important theorem later on.

section that there is no rational number whose square is equal to 2, that is, $\sqrt{2}$ is not a rational number. Thus we have line segments on the number line whose lengths cannot be expressed in rational numbers. If we wish to express the lengths of all segments on the number line as numbers, rational numbers alone are not enough for this purpose and there is need to extend our number system.

1.3 The Fact that $\sqrt{2}$ is not Rational

Let us think carefully over the question : Do we have a rational number x whose square is equal to 2 ? We know that $1^2=1$, which is less than 2. Therefore, x must be greater than 1. Also $2^2=4$, which is greater than 2. Therefore, x must be less than 2. Do we know some rational numbers between 1 and 2 ? Of course, we do. In fact, we can write as many rational numbers between 1 and 2 as we like. For example,

1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9, all lie between 1 and 2. Let us write their squares.

$$(1.1)^2 = 1.21$$

$$(1.2)^2 = 1.44$$

$$(1.3)^2 = 1.69$$

$$(1.4)^2 = 1.96$$

$$(1.5)^2 = 2.25$$

$$(1.6)^2 = 2.56$$

$$(1.7)^2 = 2.89$$

$$(1.8)^2 = 3.24$$

$$(1.9)^2 = 3.61$$

We observe that the square of none of the above numbers is equal to 2. So x is not equal to any of the above numbers. All the same we observe that $(1.4)^2=1.96$, which is less than

1 have been odd. Therefore, $p=2 \times r$ for some integer r . Substituting this value of p in $p^2=2 \times q^2$, we get

$$(2 \times r)^2 = 2 \times q^2$$

Now, $(2 \times r)^2 = (2 \times r) \times (2 \times r) = 4 \times r^2$

Thus, $4 \times r^2 = 2 \times q^2$

Dividing both sides by 2, we get

$$2 \times r^2 = q^2$$

which shows that q^2 is even. As above, this implies that q itself must be even. Therefore, $q=2 \times s$ for some integer s . We have ended up by showing that p and q are both divisible by 2. But we had assumed that p and q have no common factor. This contradiction shows that our assumption must be false and there is no rational number whose square is equal to 2. Hence $\sqrt{2}$ is not rational.

The above discussion demonstrates the inadequacy of the rational number system both from a geometric and an algebraic point of view. Geometrically, the rational numbers are not able to express all possible lengths, e. g., the length of the diagonal of a square of side 1 unit. Algebraically, the rational number system is inadequate for extracting square roots of numbers other than perfect squares. Thus there is need to 'invent' a new number system able to find answers to the above geometrical problems.

The new number system is called the real number system. It contains all the rational numbers and all the numbers known as irrational numbers. The real number system is a complete ordered field which is a continuum.

Let us next examine the properties of the real number system. Let p be an odd integer. Its square is an odd integer. Its arithmetic of rational

numbers. In particular, we know how to add two rational numbers :

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$

We can also add any number of rational numbers. For example, if we have four rational numbers, say,

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$$

Then

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$$

Let us consider the following situation where we have to keep on adding indefinitely.

Suppose an insect is trying to climb up a pole 1 metre high. Suppose further that the insect climbs $\frac{1}{2}$ metre in the first hour, $\frac{1}{4}$ metre in the second hour, $\frac{1}{8}$ metre in the third hour, $\frac{1}{16}$ metre in the fourth hour, $\frac{1}{32}$ metre in the fifth hour, and so on. During every hour, the insect climbs only half the height it climbed in the preceding hour. Will the insect reach the top of the pole? Of course, **not**. Because, for however many hours he may keep climbing, he will still be some distance from the top. However, after the first few hours, it will have reached fairly close to the top. For example, after four hours, it has climbed

$$\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \right) \text{ metres} = \frac{15}{16} \text{ metres}$$

and is only $\frac{1}{16}$ metres away from the top.

After five hours, it has climbed

$$\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} \right) \text{ metres} = \frac{31}{32} \text{ metres}$$

and is only $\frac{1}{82}$ metres away from the top.

What we are doing is adding indefinitely the rational numbers

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \dots$$

where each rational number is half of its predecessor. We express this addition by writing

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots$$

It is an example of an *infinite series*. We observe that this infinite series has the property that if we add more and more terms, then we get numbers closer and closer to 1. We express this fact by saying that the infinite series *converges* to 1. We have actually come across such ideas in Class VII.

We have learnt how to represent a rational number as a terminating or a non-terminating recurring decimal. For example, we know that

$$\frac{1}{4} = 0.25$$

and $\frac{1}{3} = 0.333\dots$

When we write $\frac{1}{4} = 0.25$, we are saying (recall the place-value system) that

$$\frac{1}{4} = \frac{2}{10} + \frac{5}{100}$$

Let us consider, however, what we mean by

$$\frac{1}{3} = 0.333\dots$$

Now $0.333\dots$ stands for the infinite series

$$\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \dots$$

and when we write

$$\frac{1}{3} = 0.333\dots$$

we really mean that the infinite series

$$\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \dots$$

converges to the rational number $\frac{1}{3}$. In other words, if we add more and more terms of the infinite series

$$\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \dots$$

we get numbers closer and closer to $\frac{1}{3}$.

The decimal representation of rational numbers is either terminating as in the case of

$$\frac{1}{4} = 0.25$$

or non-terminating and recurring as in the case of

$$\frac{1}{3} = 0.333\dots$$

This leaves out one case, namely, that of *non-terminating, non-recurring decimals*.

For example, we have the infinite series

$$\frac{1}{10} + \frac{0}{100} + \frac{1}{1000} + \frac{0}{10000} + \frac{0}{100000} + \frac{1}{1000000} + \dots$$

i.e., the decimal

$$0.101001\dots$$

This is a non-terminating, non-recurring decimal. There is no rational number to which this infinite series converges. For, remember that the decimal representation of every rational number is either terminating or non-terminating and recurring.

The non-terminating, non-recurring decimals are our new numbers, called irrational numbers.

1.5 Decimal Forms of Some Irrational Numbers

1.5.1 The Irrational Number $\sqrt{2}$: We have already seen that there is no rational number whose square is equal to 2.

Is there an irrational number whose square is equal to 2 ? The answer to this question is, 'Yes'. In fact, we have already calculated the first few decimal places of this number. We have shown that the square of 1.41 is less than 2 while the square of 1.42 is greater than 2. We can similarly check that

$$\begin{aligned}(1.414)^2 &< 2 < (1.415)^2 \\ (1.4142)^2 &< 2 < (1.4143)^2 \\ (1.41421)^2 &< 2 < (1.41422)^2 \\ (1.414213)^2 &< 2 < (1.414214)^2 \\ (1.4142135)^2 &< 2 < (1.4142136)^2 \text{ etc.}\end{aligned}$$

If we continue this process indefinitely, we get a number
 $x = 1.4142135\dots$

which is a non-terminating, non-recurring decimal. We note from the above process that with the addition of each digit on the right the square of the number becomes closer and closer to 2. We call this irrational number the square root of 2. We have already agreed to denote it by the symbol $\sqrt{2}$.

1.5.2 The Irrational Numbers $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$: Proceeding as in the case of $\sqrt{2}$, we can show that $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$ are also irrational numbers. We can similarly obtain their representations as non-terminating, non-repeating decimals. These are

$$\begin{aligned}\sqrt{3} &= 1.7320508\dots \\ \sqrt{5} &= 2.2360680\dots \\ \sqrt{7} &= 2.6457513\dots\end{aligned}$$

1.5.3 The Irrational Number π : The ratio of the circumference of any circle to its diameter is constant and does not depend on the radius of the circle. This ratio is denoted by

π . We will learn in detail about π in part II of this book. Its decimal representation is

3.141592...

1.6 The Fundamental Operations on Irrational Numbers

1.6.1 Addition and Multiplication : We have introduced positive irrational numbers such as $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, etc., to extract square roots of all positive rational numbers, whether perfect squares or not, the latter having irrational square roots. We have also seen other examples of irrational numbers. In order to be able to work with them, we introduce the operations of *addition* and *multiplication* with the same rules as for addition and multiplication with the rationals. For example, *the commutative and associative laws of addition and multiplication hold for positive irrationals also.*

Thus, if a , b , c are three positive irrationals, then

$$a + b = b + a, \quad a \times b = b \times a \quad \text{Commutative law}$$

$$(a + b) + c = a + (b + c), \quad (a \times b) \times c = a \times (b \times c) \quad \text{Associative law}$$

The distributive law also holds, i.e.,

$$(a + b) \times c = a \times c + b \times c, \quad c \times (a + b) = c \times a + c \times b$$

We note that we have used the same symbols for addition and multiplication among irrationals as for rationals.

By analogy with rational numbers, we also introduce the *relations of order*, that is, the relations of '*greater than*' and '*less than*' among the positive irrationals and use the same symbols ' $>$ ' and ' $<$ ' respectively for these relations as for the rationals.

How
 quotient *and Negative Irrationals* : We next intro-
 is a pⁿ of subtraction for irrationals as the inverse
 e same manner as for rationals. In order that
 e to subtract one irrational number from
 s, it becomes necessary to introduce negative
 us we introduce numbers such as $-\sqrt{2}$, $-\sqrt{3}$,
 ne for each of the positive irrationals, with the
 at

$$\sqrt{2} + (-\sqrt{2}) = 0$$

$$\sqrt{3} + (-\sqrt{3}) = 0$$

$$\pi + (-\pi) = 0$$

o on. As for rationals, these are called the *negatives* of
 corresponding positive irrationals, and *vice versa*.

ter introducing the negative irrationals, subtraction
 mes a particular case of addition. For, to subtract one
 onal, say a , from another irrational, say b , we can add
 negative of a to b . That is,

$$b - a = b + (-a)$$

en the operation of subtraction is considered as
 articular case of addition, it obeys the same rules as
 tion.

6.3 *Division and Reciprocals* : As for subtraction, we
 oduce the operation of division for irrational numbers as
 inverse of multiplication.

We also introduce, as for rationals, the concept of the
 iprocal of an irrational number as a number whose product
 h the given number is equal to 1. Thus, for example,

$\frac{1}{\sqrt{5}}$, $\frac{-1}{\sqrt{7}}$, $\frac{1}{\pi}$ are the reciprocals of $\sqrt{3}$, $\sqrt{5}$, $-\sqrt{7}$, π
 pectively.

We observe that we have assumed above out the arithmetical operations on rational numbers together also. For example, we can have

$$2 + \sqrt{3}, \frac{\sqrt{3}}{2}, \sqrt{5} - 1, \text{ etc.}$$

1.7 Simplification of Expressions Involving Numbers

We have learnt in Class VII that when we add or subtract two rational numbers, multiply one rational number by another, or divide one rational number by another non-zero rational number, the result is always another rational number. For example,

$$\begin{aligned} \frac{1}{2} + \frac{3}{5} &= \frac{11}{10}, & \frac{3}{4} \times \frac{6}{7} &= \frac{9}{14} \\ 7 - \frac{6}{5} &= 5\frac{4}{5}, & \frac{3}{2} \div \frac{4}{5} &= \frac{15}{8} \end{aligned}$$

In other words, any expression formed from rational numbers using the fundamental operations of arithmetic can be simplified and the result is a rational number. For example,

$$\left[\left(\frac{6}{7} \times \frac{21}{23} \right) - \frac{25}{161} \right] \div \frac{202}{161} = -\frac{1}{2}$$

However, when we form any expression from rational and irrational numbers together by using the four arithmetical operations, it is not generally possible to simplify such expressions and express the result as a simple rational number. For example, we cannot simplify $\sqrt{2} + \sqrt{3}$ further even the simple expressions such as $\frac{\sqrt{5}}{2}$, etc.

However, we like, if possible, to write expressions involving quotients of irrationals in such a way that the denominator is a positive integer. For example, we prefer to write

$$\frac{1}{\sqrt{2}} \text{ as } \frac{1 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\frac{4}{\sqrt{3}} \text{ as } \frac{4 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}} = \frac{4\sqrt{3}}{3}$$

The method that we are using here is called *rationalizing the denominator*. We give another example of rationalization of the denominator.

***Example 1 :** Simplify $\frac{2+\sqrt{5}}{3-\sqrt{5}}$

Solution : We know from Class VII, the formula

$$(a+b)(a-b) = a^2 - b^2$$

So, if we multiply $3-\sqrt{5}$ by $3+\sqrt{5}$, we will get $(3)^2 - (\sqrt{5})^2 = 9 - 5 = 4$. We, therefore, multiply both the numerator and denominator of the given fraction by $3+\sqrt{5}$. This will make the denominator a rational number. We get

$$\begin{aligned} \frac{2+\sqrt{5}}{3-\sqrt{5}} &= \frac{(2+\sqrt{5})(3+\sqrt{5})}{(3-\sqrt{5})(3+\sqrt{5})} \\ &= \frac{6+2\sqrt{5}+3\sqrt{5}+(\sqrt{5})^2}{(3)^2-(\sqrt{5})^2} \\ &= \frac{11+5\sqrt{5}}{4} = \frac{11}{4} + \frac{5\sqrt{5}}{4} \end{aligned}$$

1.8 Approximate Evaluation of Irrational Numbers

Irrational numbers are necessary for the exact solution of many mathematical problems such as extraction of square

roots, cube roots, etc., and solution of equations. But we seldom use them in our daily life and therefore we are less familiar with them. In all commercial transactions we use only rational numbers. Also all physical measurements are in terms of rational numbers. Therefore, whenever the mathematical solution of any problem of science, business, etc., involves irrational numbers, we wish to give the *final answers in terms of rational numbers. Such answers cannot be exact as no irrational number is equal to a rational number.* But, as we have seen in some cases above, we can find rational numbers as close to any irrational number as we please. This is true for all irrational numbers though we cannot prove this statement at this stage. What we do in practice is that we replace the irrational numbers by their approximate values by taking the values up to a certain number of places from their decimal representations. For example, if we use the values up to three places of decimals only

$$\sqrt{2}+1=1.414+1=2.414$$

$$\sqrt{3}-\sqrt{2}=1.732-1.414=0.318$$

$$\frac{\sqrt{3}+1}{\sqrt{2}+2} = \frac{1.732+1}{1.414+2} = \frac{2.732}{3.414} = 0.800$$

to three places of decimals. It may be emphasized that the values on the right are only approximations to the values of the expressions on the left.

In most physical problems, it is sufficient to work up to three places of decimals only. If it is required to work up to a greater number of places, we can take such values from a book of tables as explained in Unit VI.

EXERCISE 1.2

- *1. Prove that $\sqrt{2}$ is not a rational number.
2. Following the procedure used for $\sqrt{2}$, obtain the first four digits in the decimal representation of $\sqrt{3}$.
3. Write 10 irrational numbers.
4. Which of the following numbers are rational and which are irrational :
(i) $\frac{2}{3}$, (ii) $\frac{2}{3} + \sqrt{5}$, (iii) $\frac{2}{7} - \sqrt{3}$, (iv) $3 - \sqrt{3}$
(v) 0.202002000 ... (the number of zeros between consecutive 2's is increasing indefinitely).
5. Calculate $1.12057 + \sqrt{5}$ up to 5 places of decimals.
6. Taking the values of $\sqrt{2}$ and $\sqrt{3}$ up to 2 places of decimals, obtain the value of $\sqrt{2} \times \sqrt{3}$.
7. Calculate $\frac{1}{\sqrt{2}}$ up to 3 places of decimals.
8. Evaluate $\frac{1}{\sqrt{2}} + \sqrt{3}$ up to 3 places of decimals.
- *9. Simplify : $\frac{\sqrt{3} - \sqrt{2}}{\sqrt{3} + \sqrt{2}}$
- *10. Simplify : $\frac{2}{\sqrt{5} - \sqrt{3}}$

1.9. The Concept of a Real Number

We now know two types of numbers, the *rational numbers* and the *irrational numbers*. The rational numbers have

'terminating' or 'non-terminating and recurring' decimal representations, while the irrational numbers have 'non-terminating and non-recurring' decimal representations. These two types of numbers together are known as *real numbers*. Thus real numbers are of two types : those which are rational and those which are irrational.

1.10 The Real Number Line

We have learnt how to represent rational numbers on the number line. We also saw in Sec. 1.2 that the rational numbers alone are not enough to measure exactly the lengths of all possible segments on the number line. We found that the segment OD in Fig. 1.1, whose length is equal to the diagonal of a square of side 1 unit, cannot be expressed in terms of a rational number. Thus D is not a rational point on the number line. We can construct any number of such points. Some of these are constructed in the solved examples that follow this section. Thus when all the rational numbers are represented by points on the number line, all the points on the number line are not exhausted. In other words, there are many gaps left.

The irrational numbers have been invented to fill precisely these gaps. When all the rational and irrational numbers, that is, all the real numbers have been represented on the number line, it is called the real number line.

You will learn in higher studies that when all the real numbers have been represented on the number line, there are no gaps left on the line. We will not go into the construction of an arbitrary real number on the number line and will limit ourselves to the construction of a few simple irrational numbers.

Example 2 : Represent the irrational number $\sqrt{5}$ on the number line.

Solution : We observe that $5 = 1^2 + 2^2$. So we can construct $\sqrt{5}$ as the length of the hypotenuse of a right triangle whose sides are of length 1 and 2.

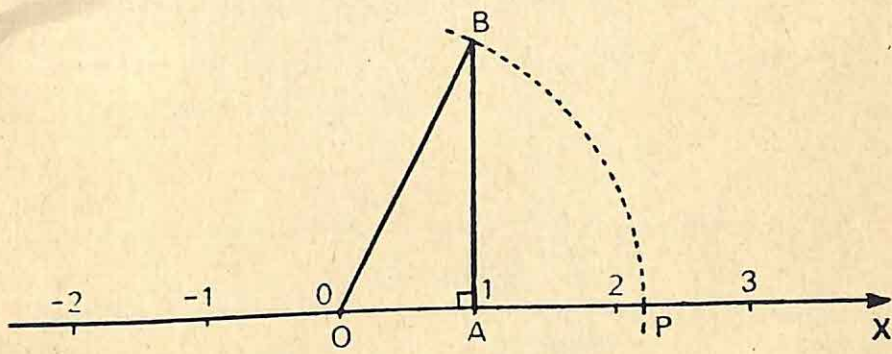


Fig. 1.2

Let OX be a number line on which O represents 0 and A represents 1. (See Fig. 1.2) Draw a line AB perpendicular to OA and mark B on it so that $AB=2$. OA. Then

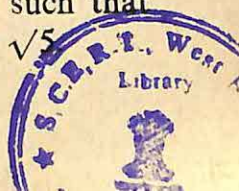
$$\begin{aligned} OB^2 &= OA^2 + AB^2 \\ &= 1 + 4 \\ &= 5 \end{aligned}$$

Now mark a point P on OX to the right of O such that $OP=OB$. Then P represents the irrational number $\sqrt{5}$.

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Date.....

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Example 3 : Represent the real number $\sqrt{10}$ on the number line.

Solution : As in Example 2, we observe that $10 = 1^2 + 3^2$.

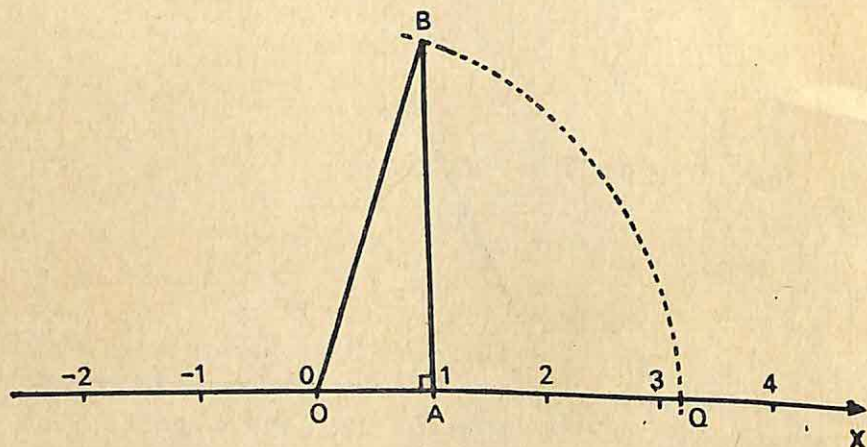


Fig. 1.3

Let OX be a number line on which O represents 0 and A represents 1. (See Fig. 1.3) Draw a line AB perpendicular to OA and mark B on it so that $AB = 3 \cdot OA$. Then

$$\begin{aligned} OB^2 &= OA^2 + AB^2 \\ &= 1 + 9 \\ &= 10 \end{aligned}$$

Now mark a point Q on OX to the right of O such that $OQ = OB$. Then Q represents $\sqrt{10}$.

EXERCISE 1.3

Represent the following numbers on the number line.

1. $\sqrt{13}$

2. $\sqrt{17}$

3. $\sqrt{18}$

4. $\frac{1}{2}\sqrt{2}$

5. $1+\sqrt{2}$

6. $\sqrt{2}-1$

- *7. Represent the number $\sqrt{3}$ on the number line.
8. Give five examples of points on the number line which are not rational.

Key Concepts**Irrational numbers****Real numbers****Arithmetical operations
on irrational numbers****Representation of real
numbers on the number
line**

and

$$(\sqrt{3})^3 = \sqrt{3} \times \sqrt{3} \times \sqrt{3} \text{ (three factors)}$$

Thus,

$$\begin{aligned} (\sqrt{3})^4 \times (\sqrt{3})^3 &= \sqrt{3} \times \sqrt{3} \times \sqrt{3} \times \sqrt{3} \text{ (4 factors)} \times \sqrt{3} \\ &\quad \times \sqrt{3} \times \sqrt{3} \text{ (3 factors)} \\ &= \sqrt{3} \times \sqrt{3} \times \sqrt{3} \times \sqrt{3} \times \sqrt{3} \times \sqrt{3} \times \sqrt{3} \\ &\quad \text{(4+3 i.e. 7 factors).} \\ &= (\sqrt{3})^7 = (\sqrt{3})^{4+3} \end{aligned}$$

Thus, we see that

$$(\sqrt{3})^4 \times (\sqrt{3})^3 = (\sqrt{3})^{4+3}$$

Can you show that

$$(\sqrt{3})^{4+3} = (\sqrt{3})^7 = 27\sqrt{3} ?$$

Example 6 : Multiply $(\sqrt{2})^3$ and $(\sqrt{2})^6$, and express the product in the exponential form.

Solution : We know that

$$(\sqrt{2})^3 = \sqrt{2} \times \sqrt{2} \times \sqrt{2} \text{ (three factors)}$$

and

$$(\sqrt{2})^6 = \sqrt{2} \times \sqrt{2} \times \sqrt{2} \times \sqrt{2} \times \sqrt{2} \times \sqrt{2} \text{ (six factors)}$$

So

$$\begin{aligned} (\sqrt{2})^3 \times (\sqrt{2})^6 &= \sqrt{2} \times \sqrt{2} \times \dots \text{ up to } 3+6=9 \text{ factors} \\ &= (\sqrt{2})^9 \\ &= (\sqrt{2})^{3+6} \end{aligned}$$

Thus, we see that

$$(\sqrt{2})^3 \times (\sqrt{2})^6 = (\sqrt{2})^{3+6}$$

Can you show that $(\sqrt{2})^{3+6} = (\sqrt{2})^9 = 16\sqrt{2} ?$

In the above two examples, the exponents were both positive. We now consider a couple of examples in which one of the exponents is negative.

Example 7 : Multiply $(\sqrt{2})^5$ and $(\sqrt{2})^{-3}$

Solution : We know that

$$(\sqrt{2})^5 = \sqrt{2} \times \sqrt{2} \times \sqrt{2} \times \sqrt{2} \times \sqrt{2} \text{ (five factors)}$$

and

$$(\sqrt{2})^{-3} = \frac{1}{(\sqrt{2})^3} = \frac{1}{\sqrt{2} \times \sqrt{2} \times \sqrt{2}} \quad (\text{Three factors in the denominator})$$

So

$$\begin{aligned} (\sqrt{2})^5 \times (\sqrt{2})^{-3} &= \sqrt{2} \times \sqrt{2} \times \sqrt{2} \times \sqrt{2} \times \sqrt{2} \\ &\quad \times \frac{1}{\sqrt{2} \times \sqrt{2} \times \sqrt{2}} \\ &= \frac{\sqrt{2} \times \sqrt{2} \times \sqrt{2} \times \sqrt{2} \times \sqrt{2}}{\sqrt{2} \times \sqrt{2} \times \sqrt{2}} \end{aligned}$$

(Five factors in the numerator and three in the denominator).

We can cancel three factors $\sqrt{2}$ each in the denominator with three factors $\sqrt{2}$ each in the numerator. We will then be left with $5-3=2$ factors in the numerator. Thus we see that

$$(\sqrt{2})^5 \times (\sqrt{2})^{-3} = \sqrt{2} \times \sqrt{2} = (\sqrt{2})^2 = (\sqrt{2})^{5-3}$$

or

$$(\sqrt{2})^5 \times (\sqrt{2})^{-3} = (\sqrt{2})^{5-3}$$

We observe that we can also write $5-3$ as $5+(-3)$. So we may also write

$$(\sqrt{2})^5 \times (\sqrt{2})^{-3} = (\sqrt{2})^{5+(-3)}$$

Can you see that $(\sqrt{2})^5 \times (\sqrt{2})^{-3} = 2$?

Example 8 : Multiply $(\sqrt{3})^4$ and $(\sqrt{3})^{-7}$

Solution : We know that

$$(\sqrt{3})^4 = \sqrt{3} \times \sqrt{3} \times \sqrt{3} \times \sqrt{3} \quad (\text{four factors})$$

and

$$(\sqrt{3})^{-7} = \frac{1}{(\sqrt{3})^7}$$

$$\begin{aligned} &= \frac{1}{\sqrt{3} \times \sqrt{3} \times \sqrt{3} \times \sqrt{3} \times \sqrt{3} \times \sqrt{3} \times \sqrt{3}} \\ &\quad (\text{Seven factors in the denominator}) \end{aligned}$$

UNIT II

Exponents and Radicals

We have learnt the terminology of exponents and studied the basic rules for multiplying and dividing numbers written in the exponential notation. Now we will study powers of positive real numbers. Also we will consider the case when the exponent is rational.

2.1 Introduction

In our earlier classes, we learnt about squares, cubes and other integral powers of rational numbers. For example, $(\frac{1}{3})^2$ is read as *second power* of $\frac{1}{3}$ or *square* of $\frac{1}{3}$ or $\frac{1}{3}$ *raised to the exponent 2* (or simply, $\frac{1}{3}$ raised to 2) or $\frac{1}{3}$ square.

Also that
$$\left(\frac{1}{3}\right)^2 = \frac{1}{3} \times \frac{1}{3}$$

Similarly
$$\left(\frac{2}{3}\right)^{-3} = \frac{1}{\left(\frac{2}{3}\right)^3} = \frac{1}{\frac{2}{3} \times \frac{2}{3} \times \frac{2}{3}}$$

The raised numeral, as we recall, is called the *exponent* or *index*. The number which is raised to some index is called the *base*. Thus in $\left(\frac{1}{3}\right)^2$, $\frac{1}{3}$ is the *base* and 2 is the *exponent*.

Similarly, in $\left(\frac{2}{3}\right)^{-3}$, $\frac{2}{3}$ is the *base* and -3 is the *exponent*.

We have now extended our number system and introduced real numbers. We, therefore, like to study powers of positive real numbers. As in the case of rational numbers, *when a real number is multiplied with itself*, we get what we call the *second power or square of the real number*.

For example, $\sqrt{2} \times \sqrt{2}$ is the square of $\sqrt{2}$ and is written as $(\sqrt{2})^2$. Higher powers of real numbers are defined similarly.

$$\text{Thus, } (\sqrt{3})^4 = \sqrt{3} \times \sqrt{3} \times \sqrt{3} \times \sqrt{3} = 3 \times 3 = 9$$

$$(\sqrt{5})^3 = \sqrt{5} \times \sqrt{5} \times \sqrt{5} = (\sqrt{5} \times \sqrt{5}) \times (\sqrt{5}) = 5\sqrt{5}$$

In other words, if b is a positive real number and m is a natural number, then b^m denotes the product of m factors, each of which is b , i.e.,

$$b^m = b \times b \times \dots \times b \text{ (} m \text{ factors)}$$

We define, as in the case of rational numbers, $b^0 = 1$ when the real number b is not equal to zero.

$$\text{Thus, } (\sqrt{2})^0 = 1, (\sqrt{3})^0 = 1, (\sqrt{5})^0 = 1, \pi^0 = 1, \text{ etc.}$$

It is clear that *the first power of a real number is the number itself*.

$$\text{Thus, } (\sqrt{2})^1 = \sqrt{2}, (\sqrt{3})^1 = \sqrt{3}, \left(\frac{\sqrt{5}}{\sqrt{2}}\right)^1 = \frac{\sqrt{5}}{\sqrt{2}}, \text{ etc.}$$

$$\text{We recall that we write } 3^{-1} \text{ for } \frac{1}{3}, 3^{-2} \text{ for } \frac{1}{3^2}, \left(\frac{2}{3}\right)^{-4}$$

$$\text{for } \frac{1}{\left(\frac{2}{3}\right)^4} \text{ etc. The same way, we shall write, } (\sqrt{2})^{-1}$$

$$\text{for } \frac{1}{\sqrt{2}}, (\sqrt{2})^{-4} \text{ for } \frac{1}{(\sqrt{2})^4}, \text{ etc.}$$

Thus,

$$\begin{aligned}
 (\sqrt{3})^4 \times (\sqrt{3})^{-7} &= \sqrt{3} \times \sqrt{3} \times \sqrt{3} \times \sqrt{3} \\
 &\quad \times \frac{1}{\sqrt{3} \times \sqrt{3} \times \sqrt{3} \times \sqrt{3} \times \sqrt{3} \times \sqrt{3} \times \sqrt{3}} \\
 &= \frac{\sqrt{3} \times \sqrt{3} \times \sqrt{3} \times \sqrt{3}}{\sqrt{3} \times \sqrt{3} \times \sqrt{3} \times \sqrt{3} \times \sqrt{3} \times \sqrt{3} \times \sqrt{3}}
 \end{aligned}$$

We note that there are four factors $\sqrt{3}$ each in the numerator which cancel with four factors $\sqrt{3}$ each in the denominator. We are thus left with $7-4=3$ factors in the denominator. Therefore,

$$(\sqrt{3})^4 \times (\sqrt{3})^{-7} = \frac{1}{\sqrt{3} \times \sqrt{3} \times \sqrt{3}} = \frac{1}{(\sqrt{3})^3} = (\sqrt{3})^{-3}$$

Thus,

$$(\sqrt{3})^4 \times (\sqrt{3})^{-7} = (\sqrt{3})^{-3} = (\sqrt{3})^{-(7-4)} = (\sqrt{3})^{4-7}$$

We observe that we can write this also as

$$(\sqrt{3})^4 \times (\sqrt{3})^{-7} = (\sqrt{3})^{4+(-7)}$$

Can you see that $(\sqrt{3})^4 \times (\sqrt{3})^{-7} = \frac{1}{3\sqrt{3}}$?

We now consider the general case of the product $b^m \times b^n$, where b is a positive real number and m, n are whole numbers, then

$$b^m = b \times b \times \dots \text{up to } m \text{ factors}$$

and

$$b^n = b \times b \times \dots \text{up to } n \text{ factors}$$

So

$$\begin{aligned}
 b^m \times b^n &= (b \times b \times \dots \text{up to } m \text{ factors}) \times (b \times b \times \dots \text{up to } n \text{ factors}) \\
 &= b \times b \times \dots \text{up to } m+n \text{ factors} \\
 &= b^{m+n}
 \end{aligned}$$

Thus,

$$b^m \times b^n = b^{m+n}$$

We can verify that this result is true also when m or n or both are equal to zero.

Next, let m, n be whole numbers, with $m \geq n$. Then

$$\begin{aligned} b^m \times b^{-n} &= b^m \times \frac{1}{b^n} \\ &= \frac{\text{b} \times \text{b} \times \dots \text{upto } m \text{ factors}}{\text{b} \times \text{b} \times \dots \text{upto } n \text{ factors}} \\ &= \text{b} \times \text{b} \times \dots \text{upto } (m-n) \text{ factors.} \end{aligned}$$

We observe that since $m \geq n$, $m-n \geq 0$. Thus,

$$b^m \times b^{-n} = b^{m-n}$$

We can similarly prove that, when $m \leq n$,

$$b^m \times b^{-n} = \frac{1}{b^{n-m}} = \frac{1}{b^{-(m-n)}} = b^{m-n}$$

It should be noted that we can also write $\frac{b^m}{b^n}$ for $b^m \times b^{-n}$

Thus we have the general rule :

If b is a positive real number and m, n are whole numbers, then

$$b^m \times b^n = b^{m+n}$$

$$\frac{b^m}{b^n} = b^m \times b^{-n} = b^{m-n}$$

We note that when we multiply two exponentials with the same base, the exponents are added and when we divide one by another, the exponent of the denominator is subtracted from that of the numerator.

We assume that the above rule is true also when the exponents m and n are rational numbers.

The above rule for products of exponentials can be extended to finite number of factors. Thus

$$b^m \times b^n \times b^p \times \dots \times b^r = b^{m+n+p+\dots+r}$$

Thus, as in the case of rational numbers, if b is a real number and r a natural number, then we denote by b^{-r} the expression

$$\frac{1}{b \times b \times \dots \times b}, \text{ (r factors in the denominator)}$$

That is, $b^{-r} = \frac{1}{b^r}$

EXERCISE 2.1

1. Write the base and exponent in each of the following numbers :

(i) $(\sqrt{2})^3$ (ii) $(\sqrt{2})^2$ (iii) 2 (iv) $(a^2)^{-3}$ (v) $(a^{-3})^2$
 (vi) a^{-6} (vii) $(\sqrt{2})^0$ (viii) 1

2. Write each of the following in the exponential notation :

(i) $\sqrt{2} \times \sqrt{2} \times \sqrt{2} \times \sqrt{2}$ (ii) $\sqrt{3} \times \sqrt{3} \times \sqrt{3}$
 (iii) 4 (iv) $\frac{3}{4}$
 (v) $a^{-2} \times a^{-2} \times a^{-2}$

- *3. Write each of $\frac{1}{2}$ and $\frac{1}{3}$ in the exponential notation in two different ways.

4. Find the value of

(i) $(3a^2)^0$ (ii) $3(a^2)^0$ (iii) $(\sqrt{2})^{-4}$
 (iv) $(\sqrt{3})^5$ (v) $\frac{\sqrt{2}}{2^0}$

2.2 Numbers with Rational Numbers as Exponents

We know that $2^2=4$. We agree to write this also in the form $4^{\frac{1}{2}} = 2$. Similarly, since

$$3^2=9, \text{ we write } 9^{\frac{1}{2}} = 3;$$

$$3^3=27, \text{ we write } 27^{\frac{1}{3}} = 3,$$

and so on. In general :

If a and b are two positive real numbers and n a natural number such that $b^n=a$, then we write also $a^{\frac{1}{n}} = b$.

For example, $(\sqrt{2})^2=2$. So we can write $2^{\frac{1}{2}} = \sqrt{2}$. Similarly, we can write $5^{\frac{1}{2}} = \sqrt{5}$.

We now consider a couple of examples to learn a rule.

Example 1 : Let us consider $(\sqrt{2})^6$

Solution : We know that

$$\begin{aligned} (\sqrt{2})^6 &= \sqrt{2} \times \sqrt{2} \times \sqrt{2} \times \sqrt{2} \times \sqrt{2} \times \sqrt{2} \\ &= (\sqrt{2} \times \sqrt{2}) \times (\sqrt{2} \times \sqrt{2}) \times (\sqrt{2} \times \sqrt{2}) \\ &= 2 \times 2 \times 2 = 2^3 \\ &= 2^{\frac{6}{2}} \end{aligned}$$

But we can write $\sqrt{2}=2^{\frac{1}{2}}$

Thus, we find that $(2^{\frac{1}{2}})^6 = 2^{\frac{1}{2} \times 6} = 2^{\frac{6}{2}}$

Example 2 : Let us consider $(\sqrt{3})^{-4}$

Solution : We know that

$$\begin{aligned} (\sqrt{3})^{-4} &= \frac{1}{(\sqrt{3})^4} \\ &= \frac{1}{(\sqrt{3} \times \sqrt{3}) \times (\sqrt{3} \times \sqrt{3})} \end{aligned}$$

$$= \frac{1}{3 \times 3} = \frac{1}{3^2} = \frac{1}{3^{\frac{4}{2}}}$$

$$= 3^{-\frac{4}{2}}$$

But we can write $\sqrt{3} = 3^{\frac{1}{2}}$

Thus, we get $(3^{\frac{1}{2}})^{-4} = 3^{\frac{1}{2} \times (-4)} = 3^{-\frac{4}{2}}$

The above examples illustrate the following rule :

If a is a positive real number and m and n integers with n positive, then

$$(a^n)^m = a^{\frac{m}{n}}$$

In other words, *the m th power of $a^{\frac{1}{n}}$ is $a^{\frac{m}{n}}$*

We now consider a few examples making use of this rule.

Example 3 : Find the value of $(8)^{\frac{2}{3}}$

Solution : By the preceding rule, $(8)^{\frac{2}{3}} = (8^{\frac{1}{3}})^2$

Now $2^3 = 2 \times 2 \times 2 = 8$ shows that $8^{\frac{1}{3}} = 2$. Thus

$$8^{\frac{2}{3}} = (8^{\frac{1}{3}})^2 = 2^2 = 2 \times 2 = 4$$

Example 4 : Find the value of $(16)^{\frac{3}{4}}$

Solution : By the same rule, $(16)^{\frac{3}{4}} = (16^{\frac{1}{4}})^3$

Also, $2^4 = 2 \times 2 \times 2 \times 2 = 16$ and so $(16)^{\frac{1}{4}} = 2$

Thus,

$$(16)^{\frac{3}{4}} = (16^{\frac{1}{4}})^3 = 2^3 = 2 \times 2 \times 2 = 8$$

EXERCISE 2.2

1. Find the value of

$$(i) (64)^{\frac{5}{6}}$$

$$(ii) (32)^{\frac{2}{5}}$$

$$(iii) (27)^{-\frac{2}{3}}$$

$$(iv) (25)^{-\frac{3}{2}}$$

$$(v) (256)^{\frac{5}{4}}$$

$$(vi) (144)^{\frac{1}{4}}$$

$$(vii) (8)^{-\frac{1}{2}}$$

$$(viii) \left(\frac{1}{4}\right)^{-\frac{3}{2}}$$

2. Find the value of

$$(i) 4 + (64)^{\frac{5}{6}}$$

$$(ii) 10 - (8)^{-\frac{1}{3}}$$

2.3 Laws of Exponents

2.3.1 We now study the multiplication of numbers written in the exponential notation with the same base which may now be a positive irrational number. We have observed in Class VII that when we multiply $(\frac{2}{3})^3$ with $(\frac{2}{3})^5$, we get the number $(\frac{2}{3})^{3+5} = (\frac{2}{3})^8$. In general, if we multiply two numbers written in the exponential notation with the same rational number as base, we get a number with the same base and an exponent which is the sum of the two exponents. We now consider some examples where the common base is irrational.

Example 5 : Multiply $(\sqrt{3})^4$ and $(\sqrt{3})^3$ and write the product in the exponential notation.

Solution : We know that

$$(\sqrt{3})^4 = \sqrt{3} \times \sqrt{3} \times \sqrt{3} \times \sqrt{3} \text{ (four factors)}$$

where b is any positive real number and $m, n, p, \dots, r$ are rational numbers.

We consider some more examples.

Example 9 : Evaluate $(\sqrt{2})^{\frac{5}{2}} \times (\sqrt{2})^{-\frac{9}{2}}$

Solution : We observe that the bases are same, therefore in multiplication, exponents have to be added. So

$$\begin{aligned} (\sqrt{2})^{\frac{5}{2}} \times (\sqrt{2})^{-\frac{9}{2}} &= (\sqrt{2})^{\frac{5}{2}-\frac{9}{2}} = (\sqrt{2})^{-\frac{4}{2}} = (\sqrt{2})^{-2} \\ &= \frac{1}{(\sqrt{2})^2} \\ &= \frac{1}{2} \end{aligned}$$

Example 10 : Evaluate $(\sqrt{5})^{-\frac{5}{2}} \times (\sqrt{5})^{-\frac{3}{2}}$

Solution : Again, the bases are same. Thus, in multiplication, exponents have to be added. We get

$$\begin{aligned} (\sqrt{5})^{-\frac{5}{2}} \times (\sqrt{5})^{-\frac{3}{2}} &= (\sqrt{5})^{-\frac{5}{2}-\frac{3}{2}} = (\sqrt{5})^{-\frac{8}{2}} = (\sqrt{5})^{-4} \\ &= \frac{1}{(\sqrt{5})^4} \\ &= \frac{1}{\sqrt{5} \times \sqrt{5} \times \sqrt{5} \times \sqrt{5}} = \frac{1}{5 \times 5} \\ &= \frac{1}{25} \end{aligned}$$

EXERCISE 2.3

- Express each of the following as a product or a quotient of powers in which both a, b (assumed positive) appear only once and all exponents are positive:

(i) $\frac{3^{-2}a^{-2}b^{-3}}{3^{-3}a^{-4}b}$,

(ii) $\left(\frac{a^5}{a^{-3}}\right)^{-2}$

$$(iii) \left(\frac{a^0 b^{-1} a^{-2} b a^{-3}}{a b^{-1}} \right)^{-2} \quad (iv) \left(\frac{a^{\frac{2}{3}}}{a^{\frac{2}{3}}} \right)$$

2. Which of the following are true ?

- (i) $3^2 \times 2^3 = 3^{2+3}$ (ii) $3^2 \times 2^3 = 2^{2+3}$
- (iii) $(\sqrt{2})^3 \times 2 = (\sqrt{2})^{3+1}$
- (iv) $(\sqrt{2})^3 \times 2 = (\sqrt{2})^{3+2}$
- (v) $(\sqrt{5})^3 \times \sqrt{25} = (\sqrt{5})^{3+1}$
- (vi) $(\sqrt{5})^3 \times \sqrt{25} = (5)^{3+2}$
- (vii) $3^2 \times 2^2 = 6^2$
- (viii) $a^{-3} + a^{-3} = a^{-6}$, a positive
- (ix) $(\sqrt{2})^4 \div (\sqrt{2})^0 = (\sqrt{2})^3$
- (x) $(\sqrt{3})^3 \times (\sqrt{3})^0 = (\sqrt{3})^4$
- (xi) $a^4 \div a^6 = a^2$, a positive
- (xii) $a^4 \div a^6 = a^{-2}$, a positive
- (xiii) $a^3 \times a^{-2} = a^5$, a positive

3. Determine k in each of the following :

- (i) $(\sqrt{2})^6 \div (\sqrt{2})^3 = (\sqrt{2})^{k-1}$
- (ii) $(\sqrt{3})^5 \div (\sqrt{3})^{-4} = (\sqrt{3})^{2k+1}$
- (iii) $(\sqrt{2})^{-3} \times (\sqrt{2})^7 = 2^k$

2.3.2 We next try to discover a rule for the power of a real number written in the exponential notation.

Consider for instance,

$$[(\sqrt{2})^3]^2 = (\sqrt{2})^3 \times (\sqrt{2})^3$$

The bases are the same. Therefore, in multiplication, the exponents have to be added. We get

$$[(\sqrt{2})^3]^2 = (\sqrt{2})^{3 \times 2} = (\sqrt{2})^6 = (\sqrt{2})^{3 \times 2}$$

Let us consider some more examples.

Example 11 : Consider $(\pi^4)^5$

Solution : We know that

$$(\pi^4)^5 = \pi^4 \times \pi^4 \times \pi^4 \times \pi^4 \times \pi^4$$

Now the bases are same. Thus, in multiplication, the exponents have to be added. We get

$$(\pi^4)^5 = \pi^{4+4+4+4+4} = \pi^{20} = \pi^{4 \times 5}$$

Example 12 : Again consider

$$[(\sqrt{5})^{\frac{2}{3}}]^{-6}$$

Solution : We have learnt that,

$$\begin{aligned} [(\sqrt{5})^{\frac{2}{3}}]^{-6} &= \frac{1}{[(\sqrt{5})^{\frac{2}{3}}]^6} \\ &= \frac{1}{(\sqrt{5})^{\frac{2}{3}} \times (\sqrt{5})^{\frac{2}{3}} \times (\sqrt{5})^{\frac{2}{3}} \times (\sqrt{5})^{\frac{2}{3}} \times (\sqrt{5})^{\frac{2}{3}} \times (\sqrt{5})^{\frac{2}{3}}} \\ &= \frac{1}{(\sqrt{5})^{\frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3}}} \\ (\sqrt{5}) \quad &= \frac{1}{(\sqrt{5})^2} = (\sqrt{5})^{-2} = (\sqrt{5})^{\frac{2}{3} \times (-3)} \end{aligned}$$

The above examples illustrate the rule :

If b is a positive real number and m, n are integers, then

$$(b^m)^n = b^{mn}$$

We now consider some examples to discover another rule

Example 13 : Let us consider $2^{\frac{3}{2}}$

Solution : Now,

$$2^{\frac{3}{2}} = (2^{\frac{1}{2}})^3 = (\sqrt{2})^3 = (\sqrt{2} \times \sqrt{2}) \times \sqrt{2} = 2\sqrt{2}$$

$$(2^3)^{\frac{1}{2}} = 8^{\frac{1}{2}} = \sqrt{8} = \sqrt{4 \times 2} = 2\sqrt{2}$$

We thus see that $2^{\frac{3}{2}} = (2^3)^{\frac{1}{2}}$

Example 14 : Again consider $(27)^{\frac{2}{3}}$

Solution : Now,

$$(27)^{\frac{2}{3}} = (27^{\frac{1}{3}})^2$$

Also $27 = 3 \times 3 \times 3 = 3^3$. Thus, $(27)^{\frac{1}{3}} = 3$. We therefore get

$$(27)^{\frac{2}{3}} = (3)^2 = 9$$

$$\begin{aligned} \text{Again } \{(27)^2\}^{\frac{1}{3}} &= \{27 \times 27\}^{\frac{1}{3}} \\ &= \{3^3 \times 3^3\}^{\frac{1}{3}} = \{3^6\}^{\frac{1}{3}} \\ &= \{3^{2+2+2}\}^{\frac{1}{3}} \\ &= \{3^2 \times 3^2 \times 3^2\}^{\frac{1}{3}} \\ &= 3^2 = 9 \end{aligned}$$

We thus see that

$$\{(27)^2\}^{\frac{1}{3}} = (27)^{\frac{2}{3}}$$

The above examples illustrate the rule :

If b is a positive real number, m an integer and n a natural number, then

$$(b^m)^{\frac{1}{n}} = b^{\frac{m}{n}}$$

Example 15 : Consider $(4^{\frac{1}{2}})^{\frac{3}{2}}$

Solution : We know that $4^{\frac{1}{2}} = \sqrt{4} = 2$. Thus

$$(4^{\frac{1}{2}})^{\frac{3}{2}} = (2)^{\frac{3}{2}}$$

$$= (2^{\frac{1}{2}})^3$$

$$= (\sqrt{2})^3$$

$$4^{\frac{3}{4}} = (4^3)^{\frac{1}{4}} = (64)^{\frac{1}{4}} = (16 \times 4)^{\frac{1}{4}}$$

$$= \{(2\sqrt{2})^4\}^{\frac{1}{4}}$$

$$= 2\sqrt{2}$$

$$= (\sqrt{2})^3$$

Thus, we get $(4^{\frac{1}{2}})^{\frac{3}{2}} = 4^{\frac{3}{4}}$

The above example illustrates the rule :

If b is a positive real number and r, s are rational numbers, then

$$(b^r)^s = b^{rs}$$

Example 16 : Find a , if $\{(\sqrt{2})^3\}^{\frac{5}{2}} = (\sqrt{2})^{2a+1}$

Solution : Using the above rule, we get,

$$\text{L.H.S.} = \{(\sqrt{2})^3\}^{\frac{5}{2}} = (\sqrt{2})^{\frac{3 \times 5}{2}} = (\sqrt{2})^{\frac{15}{2}}$$

$$\text{Thus, } (\sqrt{2})^{\frac{15}{2}} = (\sqrt{2})^{2a+1}$$

$$\text{Therefore, } \frac{15}{2} = 2a+1 \text{ or } 15 = 4a+2, \text{ i.e.,}$$

$$4a = 13$$

Thus,

$$a = \frac{13}{4}$$

EXERCISE 2.4

1. Which of the following are true and why ?
(a is always positive)

(i) $(a^3 \times a)^{-2} = a^{-7}$

(ii) $(a^3 \times a)^2 = a^7$

(iii) $(a^2 \times a^{-1})^2 = a^3$

(iv) $(a^4 \times a^{-1})^2 = a^6$

$$(v) (\sqrt{2})^3 \times (\sqrt{2})^{-5} = (\sqrt{2})^{-15}$$

$$(vi) (\sqrt{2})^3 \times (\sqrt{2})^{-5} = \frac{1}{2}$$

2. Evaluate and express the result in exponential notation.

$$(i) \{(\sqrt{2})^3 \times (\sqrt{2})^{-5}\}^6 \quad (ii) \{(\sqrt{2})^4 \times (\sqrt{2})^{-1}\}^5$$

$$(iii) \{(\sqrt{3})^5 \div (\sqrt{3})^2\}^{-2} \quad (iv) \left\{ \frac{(\sqrt{5})^6 \times (\sqrt{5})^{-2}}{(\sqrt{5})^{-2}} \right\}^2$$

3. If $3 \times (\sqrt{3})^a \times (\sqrt{3})^{\frac{1}{2}} = 3 \times \sqrt{3}$, find a .

4. If $2 \times (\sqrt{2})^5 \times (\sqrt{2})^{-\frac{2}{3}} = (\sqrt{2})^{a+1}$, find a .

2.4 Simplification of Radicals

We recall that $\sqrt{2} = 2^{\frac{1}{2}}$ is called square root of 2, $\sqrt{3} = 3^{\frac{1}{3}}$ is called square root of 3. Also $8^{\frac{1}{3}} = 2$ and we call it 3rd root or cube root of 8. Similarly, $(81)^{\frac{1}{4}} = 3$ is called 4th root of 81. In general :

If a is a positive real number and n a natural number, the positive real number $a^{\frac{1}{n}}$ is called n th root of a and is also denoted by $\sqrt[n]{a}$. In this symbol, which is called a radical, n is called the index of the radical. Also ' a ' is called the radicand.

Note : (i) When $n=1$, $a^{\frac{1}{n}} = a^1 = a$, and then $\sqrt[n]{a} = a$ itself.

(ii) When $n=2$, the index is not written down with the radical sign as we have already been using the ' $\sqrt{}$ ' for the square root.

We now consider two examples to discover a rule for the multiplication of radicals with the same index.

Example 17 : Evaluate $\sqrt{9} \times \sqrt{16}$.

Solution : We know that $3^2=9$.

Thus, $\sqrt{9}=3$. Also $4^2=16$

and we get $\sqrt{16}=4$.

Therefore, $\sqrt{9} \times \sqrt{16}=3 \times 4=12$.

Let us now evaluate $\sqrt{9 \times 16}$.

We know that $9 \times 16=144=(12)^2$, and so we get

$$\sqrt{9 \times 16} = 12.$$

We note that $\sqrt{9} \times \sqrt{16}=\sqrt{9 \times 16}$.

Example 18 : Evaluate $\sqrt[3]{27} \times \sqrt[3]{8}$.

Solution : We know that $27=3 \times 3 \times 3=3^3$. We therefore get

$$\sqrt[3]{27} = (27)^{\frac{1}{3}} = 3$$

Also $8=2 \times 2 \times 2=2^3$ and we get

$$\sqrt[3]{8} = (8)^{\frac{1}{3}} = 2$$

Thus, $\sqrt[3]{27} \times \sqrt[3]{8} = 3 \times 2 = 6$

We now evaluate $\sqrt[3]{27 \times 8}$

We have $27 \times 8=216=6 \times 6 \times 6=6^3$

Thus,

$$\sqrt[3]{27 \times 8} = (27 \times 8)^{\frac{1}{3}} = 6$$

We again note that $\sqrt[3]{27} \times \sqrt[3]{8} = \sqrt[3]{27 \times 8}$

We observe that when we multiply two radicals with the same index, we get a radical with the same index and radicand which is the product of the two radicands. Thus, we may say :

If a, b are positive real numbers and n is a natural number then,

$$\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{a \times b}$$

We now consider two examples which illustrate the rule for division of one radical by another with the same index.

Example 19 : Evaluate $\sqrt{25} \div \sqrt{4}$.

Solution : We have $25 = 5 \times 5 = 5^2$, and so

$$\sqrt{25} = 5$$

Also, $4 = 2 \times 2 = 2^2$, and so

$$\sqrt{4} = 2$$

$$\text{Thus, } \sqrt{25} \div \sqrt{4} = 5 \div 2 = \frac{5}{2}$$

Let us now calculate $\sqrt{\frac{25}{4}}$

We know that $\frac{25}{4} = \frac{5}{2} \times \frac{5}{2} = \left(\frac{5}{2}\right)^2$ Thus,

$$\sqrt{\frac{25}{4}} = \frac{5}{2}$$

We therefore note that

$$\sqrt{25} \div \sqrt{4} = \sqrt{25 \div 4}$$

Example 20 : Evaluate $\sqrt[3]{27} \div \sqrt[3]{125}$

Solution : $27 = 3 \times 3 \times 3 = 3^3$, and so

$$\sqrt[3]{27} = 3$$

Again, $125 = 5 \times 5 \times 5 = 5^3$, and so

$$\sqrt[3]{125} = 5$$

$$\text{Thus, } \sqrt[3]{27} \div \sqrt[3]{125} = 3 \div 5 = \frac{3}{5}$$

Let us now calculate $\sqrt[3]{27 \div 125}$

$$27 \div 125 = \frac{27}{125} = \frac{3}{5} \times \frac{3}{5} \times \frac{3}{5} = \left(\frac{3}{5}\right)^3$$

and so

$$\sqrt[3]{27 \div 125} = (27 \div 125)^{\frac{1}{3}} = \frac{3}{5}$$

Thus, $\sqrt[10]{1024} = 2$

Also, $32 = 2 \times 2 \times 2 \times 2 \times 2 = 2^5$

Thus, $\sqrt[5]{32} = 2$

Therefore, we get

$$\sqrt[10]{1024} \div \sqrt[5]{32} = 2 \div 2 = 1$$

EXERCISE 2.5

Using the rules learnt so far, write the following numbers in a simplified form :

1. $32 - \frac{1}{3}\sqrt{18}$

2. $\frac{\sqrt{18}}{\sqrt{2}}$

3. $\frac{\sqrt{100}}{\sqrt{20}}$

4. $\sqrt{11}(\sqrt{11} - \sqrt{44})$

5. $\frac{\sqrt{98} \times \sqrt{7}}{\sqrt{2}}$

6. $\frac{\sqrt{75} \times \sqrt{60} \times \sqrt{63}}{\sqrt{200}}$

7. $\frac{\sqrt{98} \times \sqrt{12} \times \sqrt{27}}{\sqrt{49} \times \sqrt{32}}$

8. $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{3}$

Simplify the following, assuming that a, b represent positive numbers.

9. $\frac{1}{\sqrt{ab}} \sqrt{a^5 b^2}$

10. $\sqrt[3]{\frac{a^2}{b^3}} \times \sqrt[3]{\frac{b^2}{b^{-1} a^{-1}}} \times \sqrt{\frac{a}{b}} \times \sqrt{\frac{a^{-1}}{b^{-1}}}$

$$11. \left\{ \sqrt[3]{a^2b} \times \frac{1}{\sqrt[3]{ab^2}} \right\}^{-3}$$

$$12. \{(\sqrt{a} \times \sqrt{b}) \div (\sqrt[3]{a} \times \sqrt[3]{b})\}^3$$

$$13. \{\sqrt{24a^3b^3} \div \sqrt{2a^2b}\}^{\frac{3}{2}}$$

$$14. \sqrt{3a} \times (\sqrt{3a} + \sqrt{27a^3})$$

$$15. \frac{\sqrt{ab} + \sqrt{2b}}{\sqrt{b}}$$

$$16. (\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})$$

Key Concepts

Exponentials with
positive, real base

Laws of exponents with
rational exponents

Radical

Radicand

Index

Laws of radicals
with the same
index

UNIT III

Algebraic Expressions

So far we have learnt algebraic expressions with rational coefficients. In this unit, we study about algebraic expressions with real coefficients. In particular, we learn how to add and subtract polynomials in one variable. We will also consider rational expressions and operations with them.

3.1 Recall

In class VII we have studied algebraic expressions with rational coefficients. In particular, we have learnt how to add and subtract polynomials in one variable.

Recall that an *algebraic expression* is a number or a combination of numbers (including literal numbers) formed by using the fundamental operations.

An algebraic expression of the form

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_nx^n \quad (1)^*$$

is called a *polynomial* in one variable x . Exponent of x in each term is a whole number and $a_0, a_1, a_2, \dots, a_n$ are real numbers.

*We note that, in (1), we have used $(n+1)$ constants $a_0, a_1, a_2, \dots, a_n$ for convenience, as the letters $a, b, c, \dots$ of the alphabet may not be sufficient in many cases. We also note that in each term of expression (1), the exponent of x is the same as the suffix of a .

The degree of a monomial in x is the exponent of x in the monomial. The degree of a polynomial is the greatest of the degrees of its various terms.

Thus the degree of the monomial $\frac{5}{2}x^3$ is 3 while the degree of the polynomial $\frac{2}{9}x - \frac{5}{3}x^5 + \frac{2}{7}x^3$ is 5.

We have learnt how to add (or subtract) polynomials with rational coefficients. We simply collect and add (or subtract) like terms. For example,

$$\begin{aligned} & \left(\frac{8}{9}x^2 - \frac{2}{5}x^4 + \frac{3}{7}x^5 \right) - \left(\frac{1}{9}x + \frac{2}{9}x^2 - \frac{3}{9}x^4 + \frac{4}{9}x^5 \right) \\ &= -\frac{1}{9}x + \left(\frac{8}{9}x^2 - \frac{2}{9}x^2 \right) + \left(-\frac{2}{5}x^4 + \frac{3}{9}x^4 \right) + \left(\frac{3}{7}x^5 - \frac{4}{9}x^5 \right) \\ &= -\frac{1}{9}x + \frac{6}{9}x^2 + \frac{-3}{45}x^4 - \frac{1}{63}x^5 \\ &= -\frac{1}{9}x + \frac{2}{3}x^2 - \frac{1}{15}x^4 - \frac{1}{63}x^5. \end{aligned}$$

In order to familiarize ourselves fully with what we have learnt in the previous class, let us work through a review exercise.

EXERCISE 3.1

1. What are the degrees of the following polynomials?

(i) $\frac{1}{3}x^9 - \frac{2}{7}x^4 + \frac{17}{19}x$

$$(ii) \frac{8}{11}x^2 - \frac{13}{17}x^5 + \frac{9}{13}x^{11} - \frac{12}{19}x^{25}$$

$$(iii) -\frac{3}{8}y - \frac{5}{6}y^3 + \frac{8}{15}y^4$$

2. Arrange the following monomials in increasing order of their degrees :

$$-\frac{8}{9}x, \frac{2}{11}, \frac{99}{100}x^7, \frac{101}{10}x^5$$

3. Add the following pairs of polynomials :

$$(i) \frac{2}{7}y^3 - \frac{1}{7}y^2 + \frac{6}{7}y, \frac{7}{8}y - \frac{5}{4}y^2 - \frac{3}{2}y^3$$

$$(ii) 6 + \frac{5}{6}z + \frac{2}{5}z^2 - \frac{80}{9}z^3, \frac{3}{5}z^2 + \frac{10}{11}z^3 + \frac{100}{9}z^5$$

4. Subtract the polynomial

$$-\frac{83}{7}y + \frac{18}{5}y^2 - \frac{6}{7}y^3$$

from the polynomial $\frac{6}{5}y^2 + \frac{1}{7}y^3 - \frac{2}{7}y^5$.

5. Write the coefficient of x in each of the following algebraic expressions :

$$\frac{7}{8}xy, \frac{9}{4}xyz, -\frac{15}{11}txz$$

6. Subtract the sum of

$$6.5x^4 + 6.4x^3 - 6.3x^2 \text{ and } 3.2x^4 + 3.1x^2 - 6.2x^3$$

from the sum of

$$8.1x^3 + 8.2x^2 \text{ and } 6.1x^4 + 3.7x^3 + 8$$

7. Write five polynomials in x , all of degree 5.

8. If we add two polynomials of the same degree, is the resulting polynomial again of the same degree or can it be of smaller or higher degree ?

9. What should be added to the polynomial $7x^2 - 5x + 6$ to get the polynomial $3x^2 - 7x + 7$?
10. What should be subtracted from the polynomial $8x^2 - 2x + 5$ to get the polynomial $10x^2 - 3x + 8$?

3.2 Polynomials with Real Coefficients : Addition and Subtraction

We now allow the coefficients to be real numbers. The rule for addition and subtraction of polynomials with real coefficients is the same as for polynomials with rational coefficients. For example, $\sqrt{2}x$ is a monomial in which the coefficient of x is $\sqrt{2}$. If we want to add this monomial and the monomial $3x$, then the result is

$$\sqrt{2}x + 3x = (\sqrt{2} + 3)x,$$

i.e., the coefficient of x in the sum is the sum of the real numbers $\sqrt{2}$ and 3 which are the coefficients of x in $\sqrt{2}x$ and $3x$ respectively. Let us have some examples of addition and subtraction of polynomials with real coefficients.

Example 1 : Add the polynomials

$$\frac{1}{3}x + \sqrt{2}x^2 - \frac{1}{\sqrt{3}}x^3$$

$$\text{and } \frac{2}{3}x - \sqrt{2}x^2 - \frac{1}{\sqrt{3}}x^3$$

$$\begin{aligned} \text{Solution : } & \left(\frac{1}{3}x + \sqrt{2}x^2 - \frac{1}{\sqrt{3}}x^3 \right) + \left(\frac{2}{3}x - \sqrt{2}x^2 - \frac{1}{\sqrt{3}}x^3 \right) \\ &= \left(\frac{1}{3}x + \frac{2}{3}x \right) + \left(\sqrt{2}x^2 - \sqrt{2}x^2 \right) + \left(-\frac{1}{\sqrt{3}}x^3 - \frac{1}{\sqrt{3}}x^3 \right) \end{aligned}$$

$$\begin{aligned}
 &= \left(\frac{1}{3} + \frac{2}{3} \right) x + \left(\sqrt{2} - \sqrt{2} \right) x^2 + \left(\frac{-1}{\sqrt{3}} - \frac{1}{\sqrt{3}} \right) x^3 \\
 &= x - \frac{2}{\sqrt{3}} x^3
 \end{aligned}$$

Example 2 : Subtract the polynomial

$$\frac{1}{3}x + \sqrt{2}x^2 - \frac{1}{\sqrt{3}}x^3$$

from the polynomial

$$\frac{2}{3}x - \sqrt{2}x^2 - \frac{1}{\sqrt{3}}x^3$$

Solution : The required polynomial is

$$\begin{aligned}
 &\left(\frac{2}{3}x - \sqrt{2}x^2 - \frac{1}{\sqrt{3}}x^3 \right) - \left(\frac{1}{3}x + \sqrt{2}x^2 - \frac{1}{\sqrt{3}}x^3 \right) \\
 &= \left(\frac{2}{3}x - \frac{1}{3}x \right) + \left(-\sqrt{2}x^2 - \sqrt{2}x^2 \right) + \left(\frac{-1}{\sqrt{3}}x^3 + \frac{1}{\sqrt{3}}x^3 \right) \\
 &= \left(\frac{2}{3} - \frac{1}{3} \right) x + \left(-\sqrt{2} - \sqrt{2} \right) x^2 + \left(\frac{-1}{\sqrt{3}} + \frac{1}{\sqrt{3}} \right) x^3 \\
 &= \frac{1}{3}x - 2\sqrt{2}x^2
 \end{aligned}$$

EXERCISE 3.2

1. Arrange the following monomials in increasing order of their degrees :

$$\sqrt{2}x^5, -\frac{1}{3}x^4, \frac{3}{17}x^{11}, \frac{6}{1.5}x^7$$

2. Write the coefficient of x in the following algebraic expressions :

$$(1.2) ax, \frac{1}{-\sqrt{3}}bx, \frac{\sqrt{2}}{\sqrt{5}}cdx$$

3. Add the following pairs of polynomials :

$$(i) \frac{6}{5}x - \frac{2}{\sqrt{7}}x^2 + \frac{1}{3}x^3, \sqrt{5} + \frac{1}{3}x^2 - 1.2x^3$$

$$(ii) -\frac{1}{7}y + \frac{2}{\sqrt{7}}y^2 + \sqrt{11}y^4, 8 - y^{11} - \frac{1}{\sqrt{7}}y^2$$

4. Subtract the polynomial $y^{99} - \frac{1}{3}y^{98} + \sqrt{17}$

$$\text{from } \frac{1}{\sqrt{3}}y^{99} + \frac{2}{\sqrt{3}}y^{98} - 6$$

5. Simplify :

$$\left(1 + \frac{1}{2}x + \frac{1}{\sqrt{2}}x^2\right) + \left(3 - \frac{2}{9}x + \sqrt{2}x^2\right) - \left(2 + \frac{1}{9}x - 2\sqrt{2}x^2\right)$$

6. Simplify :

$$\left(\pi x^2 + \sqrt{\frac{3}{5}}x\right) - \left(\sqrt{3} + \frac{\pi}{3}x + \sqrt{\frac{6}{5}}x^2\right)$$

3.3 Multiplication of Polynomials

Recall that if n is a natural number and x is a literal number, then x^n stands for the algebraic expression $x \times x \times \dots \times x$ to n factors.

Also $x^0 = 1$ by convention.

If x^m and x^n are two monomials. (here m and n are whole numbers), then we define the product $x^m \times x^n$ as x^{m+n} . For example :

$$x^3 \times x^4 = x^{3+4} = x^7$$

$$x^2 \times x^{10} = x^{2+10} = x^{12}$$

$$x^6 \times x^0 = x^{6+0} = x^6$$

EXERCISE 3.3

1. Carry out the following multiplications :

(i) $x^4 \times x^7$

(ii) $x^2 \times x^6$

(iii) $x^0 \times x^3$

(iv) $x^6 \times x^{14}$

2. Fill in the blanks

(i) $x^2 \times x^3 = \dots$

(ii) $x^2 \times \dots = x^8$

(iii) $x^6 \times \dots = x^6$

(iv) $x^0 \times \dots = x^5$

3.3.1 Product of two Monomials : Let ax^m and bx^n be two monomials with real coefficients a and b , then the product of ax^m and bx^n is taken to be $(a \times b) x^{m+n}$

That is, $(ax^m) \times (bx^n) = (a \times b)x^{m+n}$

Note that the coefficient of x^{m+n} in the product is the product of the coefficients of the monomials. The exponent $m+n$ of x in the product is the sum of the exponents of x in the two factors.

Example 3 : Multiply $2x^3$ and $\frac{1}{3}x^7$

Solution : By definition

$$\begin{aligned}(2x^3) \times \left(\frac{1}{3}x^7\right) &= \left(2 \times \frac{1}{3}\right)x^{3+7} \\ &= \frac{2}{3}x^{10}\end{aligned}$$

Example 4 : Multiply $\frac{-1}{\sqrt{7}}x^5$ and $\frac{10}{11}x^{13}$

Solution :

$$\begin{aligned}\left(\frac{-1}{\sqrt{7}}x^5\right) \times \left(\frac{10}{11}x^{13}\right) \\ = \left(\frac{-1}{\sqrt{7}} \times \frac{10}{11}\right)x^{5+13} \\ = \frac{-10}{11\sqrt{7}}x^{18}\end{aligned}$$

EXERCISE 3.4

Carry out the following multiplications :

$$1. \left(\frac{1}{2}x^4\right) \times \left(\frac{3}{4}x^4\right)$$

$$2. \left(\frac{\sqrt{3}}{7}x^0\right) \times \left(\frac{\sqrt{5}}{6}x^8\right)$$

$$3. \left(\frac{-1}{\sqrt{6}}x\right) \times \left(\frac{1}{7}x^6\right)$$

$$4. \left(\frac{\sqrt{10}}{11}x^{11}\right) \times \left(\frac{9}{\sqrt{10}}x^4\right)$$

$$5. \left(\sqrt{2} + \frac{1}{\sqrt{2}}\right) x^{10} \times \left(\sqrt{2} - \frac{1}{\sqrt{2}}\right) x^2$$

$$6. (4 + \sqrt{2}) x^6 \times \frac{\sqrt{2}}{\sqrt{3}} x^2$$

Fill in the blanks :

$$7. (2x^5) \times \dots = 8x^{11}$$

$$8. \dots \times \left(\frac{1}{\sqrt{3}}x^0\right) = x^2$$

$$9. \left(\frac{-16x^9}{11}\right) \times \left(\frac{4}{7}x^5\right) = \dots$$

$$10. \left(\frac{\sqrt{5}}{9}x^{16}\right) \times \dots = \frac{9}{5}x^{16}$$

3.3.2 Product of a Polynomial and a Monomial : We now learn how to multiply a polynomial by a monomial.

Let ax^n be a monomial and

$$a_0 + a_1x + a_2x^2 + \dots + a_mx^m$$

be any polynomial with real coefficients $a_0, a_1, a_2, \dots, a_m$. In analogy with the distributive law for real numbers, we would like the multiplication of polynomials to be distributive with respect to addition. For this reason, we take

$ax^n \times (a_0 + a_1x + a_2x^2 + \dots + a_mx^m)$ to be equal to $(ax^n) \times a_0 + (ax^n) \times (a_1x) + (ax^n) \times (a_2x^2) + \dots + (ax^n) \times (a_mx^m)$.
 $= (a \times a_0)x^n + (a \times a_1)x^{n+1} + (a \times a_2)x^{n+2} + \dots + (a \times a_m)x^{n+m}$

Therefore, by definition

$$ax^n \times (a_0 + a_1x + a_2x^2 + \dots + a_mx^m) \\ = (aa_0)x^n + (aa_1)x^{n+1} + \dots + (aa_m)x^{n+m}$$

Example 5 : Multiply $4x$ with $\left(\frac{3}{2}x^2 + \frac{3}{4}\right)$.

$$\begin{aligned} \text{Solution : } (4x) \times \left(\frac{3}{2}x^2 + \frac{3}{4}\right) \\ &= (4x) \times \left(\frac{3}{2}x^2\right) + (4x) \times \frac{3}{4} \\ &= \left(4 \times \frac{3}{2}\right)x^{1+2} + \left(4 \times \frac{3}{4}\right)x^{1+0} \\ &= 6x^3 + 3x \end{aligned}$$

Example 6 : Multiply $\sqrt{\frac{1}{2}}x^3$ with $(3x^2 + 4x)$.

$$\begin{aligned} \text{Solution : } \left(\sqrt{\frac{1}{2}}x^3\right) \times (3x^2 + 4x) \\ &= \left(\sqrt{\frac{1}{2}}x^3\right) \times (3x^2) + \left(\sqrt{\frac{1}{2}}x^3\right) \times (4x) \\ &= \left(\sqrt{\frac{1}{2}} \times 3\right)x^{3+2} + \left(\sqrt{\frac{1}{2}} \times 4\right)x^{3+1} \\ &= \frac{3}{\sqrt{2}}x^5 + 2\sqrt{2}x^4 \\ &= \frac{3\sqrt{2}}{2}x^5 + 2\sqrt{2}x^4 \end{aligned}$$

EXERCISE 3.5

Carry out the following multiplications :

1. $(\sqrt{2}x) \times \left(\frac{1}{2}x^3 + 4x\right)$

$$2. \left(-\frac{3}{8}x \right) \times \left(4x^2 + \sqrt{\frac{2}{3}}x \right)$$

$$3. \left(\frac{1}{6}x^5 \right) \times \left(x^3 + \frac{\sqrt{8}}{11} \right)$$

$$4. \left(-\frac{10x}{11} \right) \times \left(\frac{3}{2}x^3 + \frac{7}{6} \right)$$

3.3.3 We have learnt above about the multiplication of a polynomial with a monomial. Now we learn the multiplication of two polynomials with real coefficients.

Suppose we are given two polynomials with real coefficients. For convenience, let us call them P and Q . Then P , being a polynomial, is a sum of a number of monomials. We can multiply each of these monomials with the polynomial Q . The sum of all such products is called the product of P and Q .

Example 7 : Multiply $2x+3$ by $7x-4$.

$$\begin{aligned} \text{Solution : } & (2x+3) \times (7x-4) \\ &= (2x) \times (7x-4) + 3 \times (7x-4) \\ &= (2 \times 7)x^{1+1} + (2x) \times (-4) \\ &\quad + 3 \times (7x) + 3 \times (-4) \\ &= 14x^2 - 8x + 21x - 12 \\ &= 14x^2 + 13x - 12 \end{aligned}$$

Example 8 : Multiply $a-bx$ by $a+bx$, where a and b are real numbers.

$$\begin{aligned} \text{Solution : } & (a-bx) \times (a+bx) \\ &= a \times (a+bx) + (-bx) \times (a+bx) \\ &= a^2 + a \times (bx) + (-bx) \times a + (-bx) \times (bx) \\ &= a^2 + abx - abx - b^2x^2 \\ &= a^2 - b^2x^2 \end{aligned}$$

Example 9 : Multiply

$$\frac{1}{2}x^2 + \frac{1}{3}x + 1 \text{ by } \frac{4}{5}x^4 - \frac{2}{3}x + \frac{2}{9}.$$

$$\begin{aligned} \text{Solution : } & \left(\frac{1}{2}x^2 + \frac{1}{3}x + 1 \right) \times \left(\frac{4}{5}x^4 - \frac{2}{3}x + \frac{2}{9} \right) \\ &= \frac{1}{2}x^2 \left(\frac{4x^4}{5} - \frac{2}{3}x + \frac{2}{9} \right) + \frac{1}{3}x \left(\frac{4x^4}{5} - \frac{2}{3}x + \frac{2}{9} \right) \\ &\quad + 1 \left(\frac{4x^4}{5} - \frac{2}{3}x + \frac{2}{9} \right) \\ &= \left(\frac{1}{2}x^2 \right) \times \left(\frac{4x^4}{5} \right) + \left(\frac{1}{2}x^2 \right) \times \left(-\frac{2}{3}x \right) + \left(\frac{1}{2}x^2 \right) \times \left(\frac{2}{9} \right) \\ &\quad + \left(\frac{1}{3}x \right) \times \left(\frac{4x^4}{5} \right) + \left(\frac{1}{3}x \right) \times \left(-\frac{2}{3}x \right) + \left(\frac{1}{3}x \right) \\ &\quad \times \left(\frac{2}{9} \right) + 1 \times \left(\frac{4x^4}{5} \right) + 1 \times \left(-\frac{2}{3}x \right) + 1 \times \frac{2}{9} \\ &= \frac{4}{10}x^6 - \frac{2}{6}x^3 + \frac{1}{9}x^2 + \frac{4}{15}x^5 - \frac{2}{9}x^2 \\ &\quad + \frac{2}{27}x + \frac{4}{5}x^4 - \frac{2}{3}x + \frac{2}{9} \\ &= \frac{2}{5}x^6 + \frac{4}{15}x^5 + \frac{4}{5}x^4 - \frac{1}{3}x^3 - \frac{1}{9}x^2 - \frac{16}{27}x + \frac{2}{9} \end{aligned}$$

Example 10 : Multiply $\sqrt{2} + \frac{1}{\sqrt{3}}y - y^2$ by $6 - \sqrt{5}y$.

$$\begin{aligned} \text{Solution : } & \left(\sqrt{2} + \frac{1}{\sqrt{3}}y - y^2 \right) (6 - \sqrt{5}y) \\ &= \sqrt{2} \times 6 + \sqrt{2} \times (-\sqrt{5}y) + \left(\frac{1}{\sqrt{3}}y \right) \times 6 \\ &\quad + \left(\frac{1}{\sqrt{3}}y \right) \times (-\sqrt{5}y) + (-y^2) \times 6 + (-y^2) \times (-\sqrt{5}y) \\ &= 6\sqrt{2} - \sqrt{10}y + 2\sqrt{3}y - \frac{\sqrt{5}}{\sqrt{3}}y^2 - 6y^2 + \sqrt{5}y^3 \\ &= 6\sqrt{2} + (2\sqrt{3} - \sqrt{10})y - \left(6 + \frac{\sqrt{5}}{\sqrt{3}} \right)y^2 + \sqrt{5}y^3 \end{aligned}$$

EXERCISE 3.6

Carry out the indicated operations :

1. $(x+a) \times (x+1)$

2. $\left(\frac{1}{\sqrt{2}}x^2+x\right) \times \left(\frac{1}{3}x+1\right)$

3. $(x-1) \times (x^2+x+1) + (2.5x^2+1.7x-1)$

4. $\left(x+\frac{2}{3}\right) \times (x-\sqrt{5}) - \left(8x+\frac{1}{\sqrt{11}}x^2\right)$

5. $\left(\frac{3}{4}x-\frac{13}{18}\right) \times \left(\frac{3}{4}x+\frac{13}{18}\right) + \left(\frac{7}{8}x^2+\frac{3}{4}x\right) - \left(\frac{7}{8}x-\frac{3}{4}\right)$

6. $\left(\frac{1}{3}z^2+z+1\right) \times \left(z^2-\frac{1}{2}z+\frac{1}{9}\right)$

7. $\left(\sqrt{2}+\frac{1}{\sqrt{3}}z-z^2\right) \times \left(\frac{1}{\sqrt{2}}+z\right)$

3.4 Division of Polynomials

We are already familiar with division of numbers. For example, when we want to divide 15 by 3 we ask ourselves the question : By what number we should multiply 3 to get 15 ? We find that the answer to our question is 5, and so we say

$$15 \div 3 = 5$$

Division of polynomials is also defined in a similar manner. Consider, for example, the two polynomials $14x^2+13x-12$ and $2x+3$. In example 7 of the last section, we have seen that if we multiply $2x+3$ by $7x-4$, then we get $14x^2+13x-12$.

We can, therefore, say that $14x^2+13x-12$ divided by $2x+3$ is $7x-4$. We express this by writing

$$(14x^2 + 13x - 12) \div (2x + 3) = 7x - 4^*$$

Suppose we had not carried out the multiplication of $2x+3$ and $7x-4$ in the last section, and we were asked to divide $14x^2+13x-12$ by $2x+3$.

How would we have obtained the polynomial $7x-4$? We observe that the degree of $14x^2+13x-12$ is 2 and the coefficient of the highest degree term, namely x^2 , is 14. On the other hand, the degree of $2x+3$ is 1 and the coefficient of the highest degree term, namely x , is 2. Now by what expression we should multiply $2x$ in order to get $14x^2$? Clearly $7x$. So let us multiply $2x+3$ by $7x$. We get the polynomial

$$(7x) \times (2x+3) = 14x^2 + 21x$$

If we subtract this polynomial from $14x^2+13x-12$, we get $14x^2+13x-12 - (14x^2+21x) = -8x-12$. Now $-8x-12$ and $2x+3$ both are of degree 1. If we multiply $2x+3$ by -4 , we clearly get $-8x-12$.

Thus we see that $7x-4$ is the polynomial with which we should multiply $2x+3$ in order to get $14x^2+13x-12$.

Let us have some more examples.

Example 11 : Divide the polynomial $x^2+7x+12$ by $x+4$.

Solution : The highest degree term in $x^2+7x+12$ is x^2 and its coefficient is 1. The highest degree term in $x+4$ is x and its coefficient is 1. So we multiply $x+4$ by x . We get $(x+4) \times x = x^2 + 4x$.

* We have learnt that we cannot divide any number by zero. Similarly, in the case of polynomials, division is not valid for any value of x which makes the denominator equal to zero. Thus, in the above example, division is not valid for $x = -\frac{3}{2}$ which makes the denominator $2x+3$ equal to 0.

This fact is understood in all cases of division and will not be mentioned again and again.

Subtracting x^2+4x from $x^2+7x+12$ we get

$$x^2+7x+12-(x^2+4x)=3x+12.$$

Now $3x+12=3 \times (x+4)$. Thus

$$(x+4) \times (x+3) = x^2+4x+3(x+4) = x^2+7x+12 \text{ and } (x^2+7x+12) \div (x+4) = x+3.$$

We can exhibit our calculations as follows as in long division for natural numbers.

$$\begin{array}{r} x+4 \) \ x^2+7x+12 \left(x+3 \right. \\ \underline{ 4x} \\ 3x+12 \\ \underline{3x+12} \\ 0 \end{array}$$

As in the case of numbers, we call $x^2+7x+12$, the *dividend*, $x+4$ the *divisor*, and $x+3$ the *quotient*.

In the above displayed form of the division process, we write the dividend inside the brackets $)$ and $($, and the divisor outside on the left. The first multiplier x is written outside on the right. The product x^2+4x , of $x+4$ and x , is then written under the terms x^2+7x . This is then subtracted from the dividend to give the remainder $3x+12$. The second multiplier $+3$ is then written outside on the right, and the product of $x+4$ and $+3$, i.e., $3x+12$ is written under $3x+12$. The result of subtraction is then zero, and the quotient is $x+3$.

To indicate that the products of the divisor by the successive terms in the quotient are to be subtracted from the dividend, and the successive remainders, we change the signs of the terms of the products and write these changed signs

under the original signs. When sufficient practice is obtained, this step is not necessary.

Example 12 : Divide the polynomial

$$y^5 - 2y^4 + 4y^3 - y^2 + 2y + 6 \text{ by } y^3 - y^2 + 2$$

Solution : The division process can also be displayed as follows :

$$\begin{array}{r}
 y^3 - y^2 + 2 \overline{) y^5 - 2y^4 + 4y^3 - y^2 + 2y + 6} \\
 \underline{y^5 - y^4 + 2y^3} \\
 - y^4 + 4y^3 - 3y^2 - 2y + 6 \\
 \underline{- y^4 + y^3 - 2y} \\
 3y^3 - 3y^2 + 6 \\
 \underline{3y^3 - 3y^2 + 6} \\
 0
 \end{array}$$

Thus the quotient is $y^2 - y + 3$, i.e.,

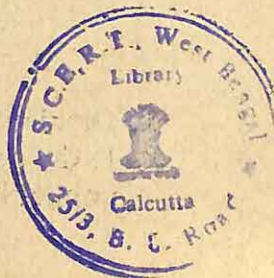
$$(y^5 - 2y^4 + 4y^3 - y^2 + 2y + 6) \div (y^3 - y^2 + 2) = y^2 - y + 3$$

In the preceding two examples, we have been able to carry out the division exactly, leaving no remainder. In such cases we say that *the dividend is exactly divisible or simply divisible by the divisor*. This, however, is not always the case.

Example 13 : Divide $4x^2 + 3x + 4$ by $2x + 1$

Solution: $2x + 1 \overline{) 4x^2 + 3x + 4} \left(2x + \frac{1}{2} \right)$

$$\begin{array}{r}
 3x + 4 \\
 \underline{4x^2 + 2x} \\
 x + 4 \\
 \underline{x + \frac{1}{2}} \\
 7\frac{1}{2}
 \end{array}$$



We note that, in this example, the division is not exact and there is a remainder $\frac{7}{2}$. This remainder is of degree 0 and we cannot carry on the division any further. We say that the quotient is $2x + \frac{1}{2}$ and the remainder is $\frac{7}{2}$.

Example 14 : Find the quotient and the remainder when $5y^3 + 7y - 6$ is divided by $y^2 + y + 1$.

Solution :

$$\begin{array}{r}
 y^2 + y + 1 \overline{) 5y^3 + 7y - 6} \quad (5y - 5 \\
 \underline{5y^3 + 5y^2 + 5y} \\
 -5y^2 + 2y - 6 \\
 \underline{-5y^2 - 5y - 5} \\
 + + + \\
 \hline
 7y - 1
 \end{array}$$

Thus the quotient is $5y - 5$ and the remainder is $7y - 1$.

Note : 1. The degree of the remainder is always less than the degree of the divisor.

2. Both the dividend and the divisor are to be arranged in descending powers of x . In general, we think of division only when the degree of the dividend is greater than or equal to that of the divisor.

EXERCISE 3.7

1. Divide the polynomial $x^2 - x - 42$ by $x - 7$
2. Is the polynomial $4y^2 - 13y - 12$ divisible by the polynomial $4y - 3$?

3. Carry out the indicated divisions :

(i) $(y^3+1) \div (y+1)$

(ii) $(y^3+1) \div (y^2-y+1)$

4. Divide $15x^4-16x^3+8x-17$ by $3x^2+x+1$ and give the quotient and remainder.

5. Divide the polynomial $2x^4+8x^3+7x^2+4x+3$ by x^2+4x+3

3.5 Rational Expressions

We have already come across examples of algebraic expressions of the type $\frac{x+1}{2x-3}$. In contrast to a polynomial, the expression $\frac{x+1}{2x-3}$ is the polynomial $x+1$ divided by the polynomial $2x-3$. Algebraic expressions of this type, which are *quotients of two polynomials*, are called *rational expressions*. These are constructed from polynomials in the same way as rational numbers are constructed from integers. Let us have some more examples of rational expressions. The algebraic expressions

$$\frac{2x-1}{3x+1}, \frac{x^2-x+1}{x^3-1}, \frac{2y+3y^2-1}{4-y+y^2}$$

are all rational expressions. The first two are rational expressions in the variable x , while the last is a rational expression in the variable y .

Note : Polynomials are particular cases of rational expressions just as integers are particular cases of rational numbers.

EXERCISE 3.8

1. Which of the following algebraic expressions are polynomials and which are rational expressions but not polynomials.

(i) $\frac{x^3-1}{x^2+2}$

(ii) $y^2 + \sqrt{2}y - 1$

(iii) $\frac{x^2 + \frac{1}{\sqrt{2}}x + 1}{x^2 - \frac{1}{\sqrt{2}}x + 1}$

(iv) $\frac{1}{3}z^2 + \frac{\sqrt{2}}{5}z$

(v) $\frac{14x^2+1}{3x-1}$

2. Write a rational expression in x whose numerator is a polynomial of degree 4 and the denominator is a polynomial of degree 3.
3. Find the difference of the numerator and denominator of the rational expression

$$\frac{3x^2 + 4x^3 - 2x + \frac{1}{2}}{5x - \frac{3}{7}x^2 + 14x^3 - 1}$$

4. Construct a rational expression whose numerator is 5 times its denominator and each is a polynomial of degree 3.

3.6 Addition of Rational Expressions

Let us now study addition of rational expressions. Do you recall how the rational numbers are added? If $\frac{a}{b}$ and

$\frac{c}{d}$ are two rational numbers, then

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$

The addition of rational expressions is also defined in a similar way.

Let us add

$$\frac{x-1}{x+2} \text{ and } \frac{2x+1}{3x-2}$$

If we follow the rule for rational numbers, then we get

$$\begin{aligned} \frac{x-1}{x+2} + \frac{2x+1}{3x-2} &= \frac{(x-1) \times (3x-2) + (x+2) \times (2x+1)}{(x+2) \times (3x-2)} \\ &= \frac{3x^2 - 2x - 3x + 2 + 2x^2 + x + 4x + 2}{3x^2 - 2x + 6x - 4} \\ &= \frac{5x^2 + 4}{3x^2 + 4x - 4} \end{aligned}$$

Rule for Addition of Rational Expressions :

If $\frac{A}{B}$ and $\frac{C}{D}$ are two rational expressions in a variable x where, A, B, C, D are polynomials in x and $B \neq 0, D \neq 0$ then $\frac{A}{B} + \frac{C}{D} = \frac{A \times D + B \times C}{B \times D}$

Let us consider some examples.

Example 15 : Add the rational expressions

$$\frac{5x-1}{5x+1} \text{ and } \frac{2x+1}{1-2x}$$

Solution :

$$\begin{aligned} &\frac{5x-1}{5x+1} + \frac{2x+1}{1-2x} \\ &= \frac{(5x-1) \times (1-2x) + (5x+1) \times (2x+1)}{(5x+1) \times (1-2x)} \end{aligned}$$

$$= \frac{5x - 10x^2 - 1 + 2x + 10x^2 + 5x + 2x + 1}{5x - 10x^2 + 1 - 2x}$$

$$= \frac{14x}{-10x^2 + 3x + 1}$$

Example 16 : Simplify :

$$\frac{2y + y^2 - 1}{1 - y} + \frac{2y - 3y^2}{1 + y}$$

Solution : $\frac{2y + y^2 - 1}{1 - y} + \frac{2y - 3y^2}{1 + y}$

$$= \frac{(2y + y^2 - 1) \times (1 + y) + (1 - y) \times (2y - 3y^2)}{(1 - y) \times (1 + y)}$$

$$= \frac{2y + 2y^2 + y^2 + y^3 - 1 - y + 2y - 3y^2 - 2y^2 + 3y^3}{1 + y - y - y^2}$$

$$= \frac{4y^3 - 2y^2 + 3y - 1}{1 - y^2}$$

EXERCISE 3.9

1. Add the following pairs of rational expressions :

(i) $\frac{x + \frac{1}{2}}{x - \frac{1}{2}}, \frac{x^2 - \frac{1}{2}}{x - \frac{1}{2}}$

(ii) $\frac{2x + x^2 - 1}{x^2 + 1}, \frac{x - x^2 + 1}{x + 2}$

(iii) $\frac{\sqrt{2}x + 1}{1 - \sqrt{2}x}, \frac{1 + \sqrt{3}x}{1 - \sqrt{3}x}$

2. Simplify :

$$\frac{y - 2y^2 + 1}{y + 2y^2 + 1} + \frac{\frac{1}{2} + y}{\frac{1}{2} - y}$$

3. Simplify

$$(i) (2x^2+1) + \frac{1}{x-1}$$

$$(ii) (1-y) + \frac{1}{1-y}$$

3.7 Subtraction of Rational Expressions

Do you recall the rule for subtraction of rational numbers? If $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers, to subtract $\frac{c}{d}$ from $\frac{a}{b}$ is the same as adding the negative (or additive inverse) of $\frac{c}{d}$ to $\frac{a}{b}$. Thus

$$\frac{a}{b} - \frac{c}{d} = \frac{a}{b} + \frac{(-c)}{d}$$

Now what should we mean by the additive inverse of a rational expression? Clearly, it should be the rational expression which when added to the given expression gives 0. For example, let us be given the rational expression $\frac{x+1}{x-1}$, then, which is the rational expression which when added to $\frac{x+1}{x-1}$ gives 0? Observe that

$$\begin{aligned} \frac{x+1}{x-1} + \left(-\frac{x+1}{x-1} \right) &= \frac{x+1}{x-1} + \frac{-x-1}{x-1} \\ &= \frac{(x+1) \times (x-1) + (x-1) \times (-x-1)}{(x-1) \times (x-1)} \end{aligned}$$

$$\begin{aligned}
 &= \frac{x^2 - x + x - 1 - x^2 - x + x + 1}{(x-1) \times (x-1)} \\
 &= \frac{0}{(x-1) \times (x-1)} \\
 &= 0
 \end{aligned}$$

Thus the additive inverse of $\frac{x+1}{x-1}$ is $\frac{-x-1}{x-1}$. In other words, it is the rational expression in which the numerator is the negative of the numerator of $\frac{x+1}{x-1}$ and the denominator is the same as that of $\frac{x+1}{x-1}$. In general,

If $\frac{P}{Q}$ is a rational expression in a variable x , then the additive inverse of $\frac{P}{Q}$ is the rational expression $-\frac{P}{Q}$.

We now define subtraction of one rational expression from another by the rule :

If $\frac{A}{B}$ and $\frac{C}{D}$ are rational expressions in a variable x , then

$$\begin{aligned}
 \frac{A}{B} - \frac{C}{D} &= \frac{A}{B} + \frac{-C}{D} \\
 &= \frac{A \times D + B \times (-C)}{B \times D} = \frac{A \times D - B \times C}{B \times D}
 \end{aligned}$$

Example 17 : Subtract the rational expression

$$\frac{x+1}{x-1} \text{ from } \frac{x-1}{x+1}$$

Solution : $\frac{x-1}{x+1} - \frac{x+1}{x-1}$

$$\begin{aligned}
 &= \frac{x-1}{x+1} + \frac{-(x+1)}{x-1} \\
 &= \frac{x-1}{x+1} + \frac{-x-1}{x-1} \\
 &= \frac{(x-1) \times (x-1) + (x+1) \times (-x-1)}{(x+1) \times (x-1)} \\
 &= \frac{x^2 - x - x + 1 - x^2 - x - x - 1}{x^2 - x + x - 1} \\
 &= \frac{-4x}{x^2 - 1}
 \end{aligned}$$

Example 18 : Simplify :

$$\frac{4y - y^2 + 1}{1 - y} - \frac{2y + y^2 - 1}{2 + y}$$

Solution :

$$\begin{aligned}
 &\frac{4y - y^2 + 1}{1 - y} - \frac{2y + y^2 - 1}{2 + y} \\
 &= \frac{(4y - y^2 + 1) \times (2 + y) - (1 - y) \times (2y + y^2 - 1)}{(1 - y) \times (2 + y)} \\
 &= \frac{8y + 4y^2 - 2y^2 - y^3 + 2 + y - (2y + y^2 - 1 - 2y^2 - y^3 + y)}{2 + y - 2y - y^2} \\
 &= \frac{8y + 4y^2 - 2y^2 - y^3 + 2 + y - 2y - y^2 + 1 + 2y^2 + y^3 - y}{2 + y - 2y - y^2} \\
 &= \frac{5y^2 + 6y + 3}{2 - y - y^2}
 \end{aligned}$$

EXERCISE 3.10

1. Subtract the rational expression $\frac{2x+3}{2x^2+x+1}$ from the rational expression $\frac{1-2x}{1+2x}$.

2. Simplify :

$$(4x - x^2 + 2) - \frac{1+x}{2+3x}$$

3. Simplify :

$$(1+8x) + \frac{8x-1}{1+8x} - \frac{2+2x}{5+2x}$$

3.8 Multiplication of Rational Expressions

How do we multiply two rational numbers ? If $\frac{a}{b}$ and $\frac{c}{d}$ are two rational numbers, recall that we define their product as $\frac{a \times c}{b \times d}$, and write

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$$

We define the product of two rational expressions $\frac{A}{B}$ and $\frac{C}{D}$ in a variable x in a similar way. Thus,

$$\frac{A}{B} \times \frac{C}{D} = \frac{A \times C}{B \times D}$$

Example 19 : Multiply the rational expressions

$$\frac{x+1}{x-1} \quad \text{and} \quad \frac{2x-1}{x-\frac{1}{2}}$$

$$\text{Solution : } \frac{x+1}{x-1} \times \frac{2x-1}{x-\frac{1}{2}} = \frac{(x+1) \times (2x-1)}{(x-1) \times \left(x-\frac{1}{2}\right)}$$

$$\begin{aligned}
 &= \frac{2x^2 - x + 2x - 1}{x^2 - \frac{1}{2}x - x + \frac{1}{2}} \\
 &= \frac{2x^2 + x - 1}{x^2 - \frac{3}{2}x + \frac{1}{2}}
 \end{aligned}$$

Example 20 : Multiply $\frac{y^2 - y + 2}{y - 3}$ by $\frac{y + 4}{y^2 + y - 1}$

Solution :

$$\begin{aligned}
 &\frac{y^2 - y + 2}{y - 3} \times \frac{y + 4}{y^2 + y - 1} \\
 &= \frac{(y^2 - y + 2) \times (y + 4)}{(y - 3) \times (y^2 + y - 1)} \\
 &= \frac{y^3 + 4y^2 - y^2 - 4y + 2y + 8}{y^3 + y^2 - y - 3y^2 - 3y + 3} \\
 &= \frac{y^3 + 3y^2 - 2y + 8}{y^3 - 2y^2 - 4y + 3}
 \end{aligned}$$

EXERCISE 3.11

1. Multiply the following rational expressions :

(i) $\frac{5x + 3}{5x - 1}$ and $\frac{2x - 1}{x + 1}$

(ii) $\frac{x + \frac{1}{2}}{x^2 + 1}$ and $\frac{2x + 4}{3}$

(iii) $\frac{3y + y^2}{18y - 1}$ and $\frac{5}{4y + 1}$

(iv) $8x + 7x^2 + 6$ and $\frac{1}{x + 1}$

2. Simplify :

$$(x+5) + \frac{4x+7}{x-1} \times \frac{7-4x^2}{x+4}$$

3. Simplify :

$$(8y-y^2+6)-(y^2+6-7y) \times \frac{y-5}{6y+5}$$

3.9 Reciprocal of a Rational Expression

We have learnt in the last section how to multiply two rational expressions in a variable x .

$$\text{What is } \frac{x-1}{2x+1} \times \frac{2x+1}{x-1} ?$$

According to our rule for multiplication, the product is

$$\frac{(x-1) \times (2x+1)}{(2x+1) \times (x-1)}$$

Even without multiplying out the numerator and denominator we can see that they are both equal (Why?). Do you remember what is $\frac{5}{5}$ or $\frac{7}{7}$ or $\frac{3}{3}$? These are all equal to 1.

This is also true of rational expressions. If the numerator and the denominator of a rational expression are both equal to the same polynomial, then we consider that rational expression is equal to 1. Thus

$$\frac{x-1}{2x+1} \times \frac{2x+1}{x-1} = 1$$

In rational numbers, you would recall that we call $\frac{b}{a}$ the reciprocal of $\frac{a}{b}$. Here also we say that

$$\frac{2x+1}{x-1} \text{ is the reciprocal of } \frac{x-1}{2x+1}.$$

Similarly,

$$\frac{x^2+x+1}{2x+3} \text{ is the reciprocal of } \frac{2x+3}{x^2+x+1}$$

$$\frac{5y^2-2y+7}{3y^2+4y+9} \text{ is the reciprocal of } \frac{3y^2+4y+9}{5y^2-2y+7}$$

EXERCISE 3.12

1. Write down the reciprocals of the following rational expressions :

(i) $\frac{0.5x+0.7}{3x+0.1}$

(ii) $\frac{8x^2+7x+0.1}{7x^2-2x+0.3}$

(iii) $\frac{20y-8y^2+5}{3y+0.8}$

2. Find the sum of the rational expression $\frac{x^2+20x+10}{x+1}$ and its reciprocal.

3. Write a rational expression in x whose reciprocal has its numerator of degree 2 and denominator of degree 3.

4. What is the reciprocal of :

(i) $2x+3$

(ii) $\frac{1}{x+1}$

3.10 Division of Rational Expressions

Recall that if we want to divide a rational number $\frac{a}{b}$

by a rational number $\frac{c}{d}$, then we multiply $\frac{a}{b}$ and the reciprocal of $\frac{c}{d}$ i.e., $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$.

We divide one rational expression by another in exactly the same way.

If $\frac{A}{B}$ and $\frac{C}{D}$ are two rational expressions in a variable, say x , then

$$\frac{A}{B} \div \frac{C}{D} = \frac{A}{B} \times \frac{D}{C} = \frac{A \times D}{B \times C}$$

Example 21 : Divide the rational expression

$$\frac{x^2+x+1}{x-1} \text{ by } \frac{x^2-1}{x+2}$$

Solution :

$$\begin{aligned} & \frac{x^2+x+1}{x-1} \div \frac{x^2-1}{x+2} \\ &= \frac{x^2+x+1}{x-1} \times \frac{x+2}{x^2-1} \\ &= \frac{(x^2+x+1) \times (x+2)}{(x-1) \times (x^2-1)} \\ &= \frac{x^3+x^2+x+2x^2+2x+2}{x^3-x^2-x+1} \\ &= \frac{x^3+3x^2+3x+2}{x^3-x^2-x+1} \end{aligned}$$

EXERCISE 3.13

1. Divide the rational expression

$$\frac{2x+1}{x-1} \text{ by } \frac{x-1}{x+1}$$

2. Simplify :

$$\frac{3y+5}{5-2y} \div \frac{y+1}{y-1}$$

3. Simplify :

$$(i) \left\{ \frac{2x^2+1}{x-1} + \frac{x-1}{x+1} \right\} \div \frac{x^2-1}{2x}$$

$$(ii) \left\{ (y+y^2+2) \times (3y-1) - \frac{1}{4} \right\} \div \frac{1}{y}$$

4. Let
- $P = \frac{x}{x+1}$
- and
- $Q = \frac{1}{x}$

Find (i) $P+Q$ (ii) $P-Q$ (iii) $P \times Q$ (iv) $P \div Q$

Key Concepts

Polynomials with real coefficients

Rational expressions

Addition and subtraction of polynomials

Addition and subtraction of rational expressions

Multiplication of polynomials

Multiplication and division of rational expressions

Division of polynomials

Reciprocal of a rational expression

UNIT IV

Special Products and Factors

In this unit we will learn the cube of a binomial and the sum and difference of the cubes of two numbers.

4.1 Cube of a Binomial

4.1.1 Recall : We have learnt in earlier classes how to find the square of a binomial. Recall that, for the binomial $x+y$,
$$(x+y)^2 = x^2 + 2xy + y^2$$

In other words, to obtain the square of a binomial $x+y$, we add the square of x , twice the product of x and y , and the square of y . We give a review exercise.

EXERCISE 4.1

Calculate the squares of the following :

- | | | |
|--------------------|------------------------------|------------|
| 1. $x+1$ | 2. $2x+3$ | 3. $y-1$ |
| 4. x | 5. 99 | 6. $2x+3z$ |
| 7. $2x+1$ | 8. $1-3x$ | 9. $2z-5$ |
| 10. $\sqrt{2} + x$ | 11. $x + \sqrt{\frac{5}{2}}$ | |

Find the products as indicated :

12. $(\sqrt{2}x+3)\left(\frac{x}{\sqrt{2}}+3\right)$

13. $\sqrt{2}d(d^2+4d)$

14. $\left(\sqrt{2}+\frac{1}{\sqrt{2}}x\right)\left(\sqrt{2}-\frac{1}{\sqrt{2}}x\right)$

4.1.2 Cube of a Sum of two Monomials : Just as we use the phrase 'square of an expression' to mean the expression multiplied with itself, we call an expression multiplied with itself twice, the cube of the expression.

For example, the cube of 2, written 2^3 , is

$$2 \times 2 \times 2 = 8,$$

$$\text{Cube of } x = x^3 = x \times x \times x$$

$$\text{Cube of } x+1 = (x+1)^3 = (x+1) \times (x+1) \times (x+1)$$

Let us calculate the cube of a binomial $x+y$ in general.

Now we already know that

$$(x+y)^2 = (x+y) \times (x+y) = x^2 + 2xy + y^2$$

Thus,

$$\begin{aligned}(x+y)^3 &= (x+y) \times (x+y)^2 \\ &= (x+y) \times (x^2 + 2xy + y^2) \\ &= x^3 + 2x^2y + xy^2 + yx^2 + 2xy^2 + y^3 \\ &= x^3 + 3x^2y + 3xy^2 + y^3\end{aligned}$$

This gives us the formula (F) for finding the cube of a binomial :

FI. $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

In words : *The cube of a binomial is equal to the cube of the first term, plus three times the square of the first term*

multiplied by the second, plus three times the first term multiplied by the square of the second, plus the cube of the second.

Example 1 : Expand $(x+1)^3$.

$$\begin{aligned}\text{Solution : } (x+1)^3 &= x^3 + 3x^2 \times 1 + 3x \times 1^2 + 1^3 \\ &= x^3 + 3x^2 + 3x + 1\end{aligned}$$

Example 2 : Expand $(2x+3y)^3$.

$$\begin{aligned}\text{Solution : } (2x+3y)^3 &= (2x)^3 + 3(2x)^2(3y) + 3(2x)(3y)^2 + (3y)^3 \\ &= 8x^3 + 3 \times 4x^2 \times 3y + 3 \times 2x \times 9y^2 + 27y^3 \\ &= 8x^3 + 36x^2y + 54xy^2 + 27y^3\end{aligned}$$

Example 3 : Calculate $(101)^3$

$$\begin{aligned}\text{Solution : } (101)^3 &= (100+1)^3 \\ &= 100^3 + 3 \times 100^2 \times 1 + 3 \times 100 \times 1^2 + 1^3 \\ &= 1000000 + 30000 + 300 + 1 \\ &= 1030301\end{aligned}$$

EXERCISE 4.2

Calculate the cubes of the following expressions :

1. $a+b$

2. $a+2b$

3. $x+2$

4. $x+2y$

5. $6x+7y$

6. $x + \frac{1}{\sqrt{3}}$

7. $3x+1$

Find the value of :

8. 102^3

9. 205^3

10. 1001^3

11. $(10.5)^3$

4.1.3 Cube of a Difference of two Monomials : We have calculated above the cube of the sum of two monomials. What about the cube of the difference of two monomials ?

We already know that

$$(x-y)^2 = (x-y) \times (x-y) = x^2 - 2xy + y^2$$

We can, therefore, calculate easily the cube of $x-y$.

$$\begin{aligned}(x-y)^3 &= (x-y) \times (x-y)^2 \\ &= (x-y) \times (x^2 - 2xy + y^2) \\ &= x^3 - 2x^2y + xy^2 - yx^2 + 2xy^2 - y^3 \\ &= x^3 - 3x^2y + 3xy^2 - y^3\end{aligned}$$

This gives us the following formula for finding the cube of a difference of two monomials :

$$\text{F II. } (x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$$

It may be observed that in the expansion of $(x-y)^3$, the terms on the right are the same as in the expansion of $(x+y)^3$, but the signs of the terms are alternately positive and negative.

Example 4 : Find $(2a-3b)^3$

$$\begin{aligned}\text{Solution : } (2a-3b)^3 &= (2a)^3 - 3 \times (2a)^2 \times 3b + 3(2a)(3b)^2 - (3b)^3 \\ &= 8a^3 - 3 \times 4a^2 \times 3b + 3 \times 2a \times 9b^2 - 27b^3 \\ &= 8a^3 - 36a^2b + 54ab^2 - 27b^3\end{aligned}$$

Example 5 : Calculate 99^3

$$\begin{aligned}\text{Solution : } 99^3 &= (100-1)^3 \\ &= 100^3 - 3 \times 100^2 \times 1 + 3 \times 100 \times 1^2 - 1^3 \\ &= 1000000 - 30000 + 300 - 1 \\ &= 1000300 - 30001 \\ &= 970299\end{aligned}$$

EXERCISE 4.3

Calculate the cubes of the following expressions :

1. $a-b$

2. $2a-5b$

3. $8x-1$

4. $1 - \frac{7x}{10}$

5. $2x-3z$

Find the value of :

6. $(97)^3$

7. $(999)^3$

8. $(9.9)^3$

4.2 Some Special Products

Sometimes the product of two polynomials turns out to be of a rather simple type. We have come across one such beautiful product in class VII, viz,

$$(x+y) \times (x-y) = x^2 - y^2$$

This product is often useful when we are working with algebraic expressions.

Let us consider the following product

$$(x+y) \times (x^2 - xy + y^2)$$

When we multiply it out, we get

$$\begin{aligned} & (x+y) \times (x^2 - xy + y^2) \\ &= x^3 - x^2y + xy^2 + yx^2 - xy^2 + y^3 \\ &= x^3 + y^3 \end{aligned}$$

Thus, we have

$$\text{F III. } (x+y)(x^2 - xy + y^2) = x^3 + y^3$$

Note that the right hand side is a sum of two cubes.

Example 6 : Find $(x+1)(x^2 - x + 1)$.

Solution : If we take $y=1$ in F III, we get at once,

$$(x+1)(x^2 - x + 1) = x^3 + 1^3 = x^3 + 1$$

Example 7 : Find $(2a+3b)(4a^2-6ab+9b^2)$

Solution : Note that the second factor

$$\begin{aligned} &4a^2-6ab+9b^2 \\ &= (2a)^2 - (2a)(3b) + (3b)^2 \end{aligned}$$

Comparing the product

$$(2a+3b) \{ (2a)^2 - (2a)(3b) + (3b)^2 \}$$

with the left hand side in F III, we see that we can apply FIII with $x=2a$, $y=3b$. We, therefore, have

$$\begin{aligned} &(2a+3b)(4a^2-6ab+9b^2) \\ &= (2a)^3 + (3b)^3 \\ &= 8a^3 + 27b^3 \end{aligned}$$

EXERCISE 4.4

Find the following products :

1. $(a+2)(a^2-2a+4)$
2. $(1-a+a^2)(1+a)$
3. $(0.7x+0.9y)(0.49x^2-0.63xy+0.81y^2)$
4. $(2x+7)\{(2x)^2-7 \times 2x+49\}$
5. $\left(\frac{x}{2} + \frac{3y}{4}\right) \left(\frac{x^2}{4} - \frac{3xy}{8} + \frac{9y^2}{16}\right)$

If we replace y by $-y$ in F III, we get

$$\text{F IV. } (x-y)(x^2+xy+y^2) = x^3 - y^3$$

We can also verify this by direct multiplication.

$$\begin{aligned} &(x-y)(x^2+xy+y^2) \\ &= x^3 + x^2y + xy^2 - yx^2 - xy^2 - y^3 \\ &= x^3 - y^3 \end{aligned}$$

Example 8 : Find $(x-1)(x^2+x+1)$

Solution : Using F IV taking $y=1$, we have

$$\begin{aligned}(x-1)(x^2+x+1) &= x^3 - 1^3 \\ &= x^3 - 1\end{aligned}$$

Example 9 : Find $(2a-5b)(4a^2+10ab+25b^2)$.

Solution : The second factor

$$4a^2+10ab+25b^2 = (2a)^2 + (2a)(5b) + (5b)^2$$

keeping this in mind and then comparing the product

$$(2a-5b)(4a^2+10ab+25b^2)$$

with the left hand side of F IV, we see that the formula is applicable with $x=2a$, $y=5b$.

Applying F IV we have

$$\begin{aligned}(2a-5b)(4a^2+10ab+25b^2) &= (2a)^3 - (5b)^3 \\ &= 8a^3 - 125b^3\end{aligned}$$

EXERCISE 4.5

Find the following products :

- $(1-x)(1+x+x^2)$
- $(25x^2+15xy+9y^2)(5x-3y)$
- $(2a-1)(1+2a+4a^2)$
- $(1-2a)(4a^2+1+2a)$
- $\left(\frac{1}{49}y^2 + \frac{3}{7}xy + 9x^2\right) \left(3x - \frac{1}{7}y\right)$

4.3 Review of Factoring Techniques Covered in Class VII

The operation of expressing a polynomial as a product of two (or more) polynomials of smaller degrees is called

factorization. We have learnt four factoring methods for quadratic expressions with rational coefficients in Class VII. The methods continue to be valid when the coefficients are real numbers. Let us recall these methods.

I. *If all terms of a polynomial have a common factor, we can express the polynomial as a product of two factors, one of which is the common factor.*

Example 10 : $15x^2 + 3x$ is a polynomial in which each term is a multiple of $3x$. We can write

$$(15x^2 + 3x) = 3x \times (5x + 1)$$

$$\text{II. } x^2 - y^2 = (x + y)(x - y)$$

That is, *the difference of the squares of two variables is the product of the sum and difference of these two variables.*

$$\begin{aligned} \text{Example 11 : } 49x^2 - 25y^2 &= (7x)^2 - (5y)^2 \\ &= (7x + 5y)(7x - 5y) \end{aligned}$$

$$\text{III. } x^2 + 2xy + y^2 = (x + y)^2$$

$$\begin{aligned} \text{Example 12 : } 4x^2 + 20xy + 25y^2 &= (2x)^2 + 2(2x)(5y) + (5y)^2 \\ &= (2x + 5y)^2 \end{aligned}$$

$$\text{IV. } x^2 - 2xy + y^2 = (x - y)^2$$

$$\begin{aligned} \text{Example 13 : } x^2 - 2\sqrt{3}xy + 3y^2 \\ &= (x)^2 - 2(x)(\sqrt{3}y) + (\sqrt{3}y)^2 \\ &= (x - \sqrt{3}y)^2 \end{aligned}$$

EXERCISE 4.6

Factorize :

1. $\sqrt{2}x + 2x^2$

3. $x^2 - 2$

2. $7y + 21y^2 - 49y^3$

4. $7y^2 - 3z^2$

5. $y^2 + 2 \times 2y + 4$

6. $0.01x^2 + 0.2xy + y^2$

7. $x^2 - 2\sqrt{3}x + 3$

8. $2x^2 - 2\sqrt{2}xy + y^2$

4.4 Factorization of a Second Degree Trinomial

A second degree trinomial in a variable x is a polynomial of the form

$$ax^2 + bx + c$$

where a, b, c are (known) real numbers. This is also called a *quadratic expression*. To give a method of factorizing such polynomials in some simple cases, let us consider the product

$$(x+a)(x+b)$$

On carrying out the multiplication, we get

$$(x+a)(x+b) = x^2 + (a+b)x + ab$$

Thus a polynomial of the form

$$x^2 + (a+b)x + ab$$

can be factorized as

$$(x+a)(x+b)$$

$$\text{F V. } x^2 + (a+b)x + ab = (x+a)(x+b)$$

Note that if $x^2 + Ax + B$ is an arbitrary quadratic expression, where the coefficient of x^2 is 1, then it is of the form $x^2 + (a+b)x + ab$, only if we can find two real numbers a and b , whose sum $a+b$ is A and whose product ab is B .

Example 14 : Factorize $x^2 + 7x + 12$.

Solution : If we can think of two numbers whose sum is 7 and product is 12, then FV can be used. Now 12 can be

factored as $1 \times 12, 2 \times 6, 3 \times 4$. Of these the factors 3 and 4 are such that their sum is 7. Thus we can write

$$x^2 + 7x + 12 = x^2 + (4+3)x + 4 \times 3$$

and so

$$x^2 + 7x + 12 = (x+4)(x+3)$$

Example 15 : Factorize $x^2 + 2x - 8$.

Solution : We have to think of two numbers whose product is -8 and sum is 2. The factors of -8 are

$$(-1) \times 8, (-2) \times 4, 1 \times (-8), \text{ and } 2 \times (-4).$$

Evidently, the sum of factors -2 and 4 is $+2$. So,

$$\begin{aligned}x^2 + 2x - 8 &= x^2 + (4-2)x + 4 \times (-2) \\&= x^2 + 4x - 2x - 2 \times 4 \\&= x(x+4) - 2(x+4) \\&= (x+4)(x-2)\end{aligned}$$

EXERCISE 4.7

Factorize :

1. $x^2 + 2x + 1$

2. $x^2 - 10x + 25$

3. $x^2 + 6x + 5$

4. $y^2 - 2y - 3$

5. $z^2 - 5z + 6$

6. $y^2 + \frac{7}{12}y + \frac{1}{12}$

7. $x^2 + x - 20$

8. $x^2 + 3ax + 2a^2$

9. $x^2 + 2kx - 3k^2$

10. $x^2 - px - 6p^2$

4.5 Factorization of the Sum and Difference of two Cubes

We have learnt in Section 4.2 that

$$(x+y)(x^2-xy+y^2)=x^3+y^3$$

In other words, if we want to factorize the sum of two cubes x^3+y^3 , then we can say that $x+y$ and x^2-xy+y^2 are factors of x^3+y^3 . Thus

$$\text{F VI. } x^3+y^3=(x+y)(x^2-xy+y^2)$$

Suppose we want to factorize the polynomial x^3+1 .

We note that $x^3+1=x^3+1^3$. We can, therefore, use F VI and get the factorization,

$$x^3+1=(x+1)(x^2-x+1)$$

Can you factorize the polynomial x^2-x+1 ? Do try.

Note : Given any quadratic expression x^2+Ax+B , it is not always possible to find two real numbers whose sum is A and product is B. Thus we cannot apply FV always.

Example 16 : Factorize $8a^3+27b^3$

Solution : We note that $8a^3+27b^3$

$$=(2a)^3+(3b)^3$$

i.e., the given expression is a sum of two cubes. We use FVI and get

$$\begin{aligned} 8a^3+27b^3 &= (2a+3b)\{(2a)^2-(2a)(3b)+(3b)^2\} \\ &= (2a+3b)(4a^2-6ab+9b^2) \end{aligned}$$

Example 17 : Factorize $2\sqrt{2}x^3+125$

Solution : We note that

$$2\sqrt{2}x^3+125=(\sqrt{2}x)^3+(5)^3$$

$$\begin{aligned} \text{Or, } 2\sqrt{2}x^3+125 &= (\sqrt{2}x+5)\{(\sqrt{2}x)^2-(\sqrt{2}x)(5)+(5)^2\} \\ &= (\sqrt{2}x+5)(2x^2-5\sqrt{2}x+25) \end{aligned}$$

In Section 4.2 we also learnt that

$$(x-y)(x^2+xy+y^2)=x^3-y^3$$

Therefore, whenever an expression is a difference of two

cubes such as $x^3 - y^3$, then we can factorize it as $(x - y)(x^2 + xy + y^2)$.

$$\text{F VII } x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$

Example 18 : Factorize $1 - a^3$.

Solution : $1 - a^3 = 1^3 - a^3$ which is a difference of two cubes. Using F VII we have

$$\begin{aligned} 1 - a^3 &= 1^3 - a^3 \\ &= (1 - a)(1 + a + a^2) \end{aligned}$$

Example 19 : Factorize $125x^3 - 8y^3$.

Solution : We note that

$125x^3 - 8y^3 = (5x)^3 - (2y)^3$ is a difference of two cubes.

We can therefore apply F VII and get

$$\begin{aligned} 125x^3 - 8y^3 &= (5x)^3 - (2y)^3 \\ &= (5x - 2y) \{ (5x)^2 + (5x)(2y) + (2y)^2 \} \\ &= (5x - 2y)(25x^2 + 10xy + 4y^2) \end{aligned}$$

*4.6 Identities and Conditional Identities

Let us consider the formula

$$(x + y)^2 = x^2 + 2xy + y^2 \quad (1)$$

Let us give any values to x and y , say $x=1$, $y=1$, then

$$\text{L.H.S. of (1)} = (1+1)^2 = 4,$$

$$\text{R.H.S. of (1)} = 1^2 + 2 \times 1 \times 1 + 1^2 = 4,$$

and are thus equal. Let us try some other pairs of values for x and y , say $x=1$, $y=2$; $x=2$, $y=1$; $x=3$, $y=-5$, and so on. In each case we find that the values of the L.H.S. and R.H.S. are the same. You may try any other pairs of values. We thus find that (1) is satisfied for all values of x and y . We call such an equation an *identity*. Thus

An identity is an equation in any number of unknowns which is true for all values of the unknowns.

We have met with several examples of identities in two unknowns. All the equations in Formula F I. to F IV are identities.

Now let us consider the product

$$\begin{aligned}(x+y+z)(x^2+y^2+z^2-yz-zx-xy) \\ = x^3+xy^2+xz^2-xyz-x^2z-x^2y \\ +yx^2+y^3+yz^2-y^2z-yzx-xy^2 \\ +zx^2+zy^2+z^3-yz^2-z^2x-zxy \\ = x^3+y^3+z^3-3xyz\end{aligned}$$

as all the other terms in the product cancel out.

Thus,

$$\begin{aligned}(x+y+z)(x^2+y^2+z^2-yz-zx-xy) \\ = x^3+y^3+z^3-3xyz\end{aligned}\quad (2)$$

Since we have obtained equation (2) without assuming any particular values for x , y , z , it is true whatever values we may give to x , y , z . Thus (2) is also an identity.

Next, let us assume that x , y , z have such values, that $x+y+z=0$. For example, $x=1$, $y=2$, $z=-3$. Then the factor $x+y+z$ on the L.H.S of (2) is zero, i.e., the L.H.S of (2) is zero. Therefore the R.H.S. of (2) must also be zero. We therefore conclude that :

If $x+y+z=0$ then

$$x^3+y^3+z^3-3xyz=0, \quad (3)$$

or,

$$x^3+y^3+z^3=3xyz \quad (3')$$

We note that equation (3) or (3') is not true for all values of x , y , z . For example, if we take $x=1$, $y=1$, $z=2$ in (3'), the L.H.S. $=1+1+8=10$, while the R.H.S. $=3 \times 1 \times 1 \times 2=6$. Thus equation (3') is not satisfied for $x=1$, $y=1$, $z=2$.

Equation (3) or (3') is true for such values of x , y , z which satisfy the equation $x+y+z=0$ or, in other words, which satisfy the condition $x+y+z=0$. For this reason, (3) or (3') is called a *conditional identity*.

Similarly, if we take

$$x^2 + y^2 + z^2 - yz - zx - xy = 0$$

in (2), then the L.H.S. of (2) is again 0 and so the R.H.S. is also zero. We then get a second conditional identity.

'If $x^2 + y^2 + z^2 - yz - zx - xy = 0$,

then $x^3 + y^3 + z^3 = 3xyz$ ' (4)

An example of such a set of values of the unknowns is $x=1$, $y=1$, $z=1$.

Another example, a rather easy one, of a conditional identity is

'If $x+y=0$ then $x^2-y^2=0$ ' (5)

Can you show that we also have,

'If $x-y=0$, then $x^2-y^2=0$?' (6)

We say that :

A conditional identity is an equation in two or more unknowns which holds only when the unknowns are subject to some condition.

We now solve an example on the use of the conditional identity (3').

Example 20 : Prove that, for all real numbers a , b , c ,

$$(b-c)^3 + (c-a)^3 + (a-b)^3 = 3(b-c)(c-a)(a-b)$$

Solution : Let us write $b-c=x$, $c-a=y$, $a-b=z$, then

$$x+y+z=b-c+c-a+a-b=0.$$

Now by equation (3') above,

$$\begin{aligned} (b-c)^3 + (c-a)^3 + (a-b)^3 &= x^3 + y^3 + z^3, \text{ where } x+y+z=0 \\ &= 3xyz \\ &= 3(b-c)(c-a)(a-b) \end{aligned}$$

EXERCISE 4.8

Factorize :

1. $a^3 - 1$

2. $a + a^4$

3. $8x^3 + 343y^3$

4. $\frac{1}{216} a^3 + 8b^3$

5. $\frac{1}{5\sqrt{5}} x^3 - 2\sqrt{2} y^3$

6. $0.001x^3 - 0.125y^3$

*7. Prove that $a^3(b-c)^3 + b^3(c-a)^3 + c^3(a-b)^3$
 $= 3abc(a-b)(b-c)(c-a)$

Key Concepts

Cube of a binomial

Factoring a second degree trinomial

Cube of the sum and difference of two numbers

Identity

Sum and difference of cubes of two numbers

Conditional identity

Miscellaneous Exercise I

(On Units I, II, III and IV)

1. Find a rational number between 1 and $\sqrt{2}$.
2. Find an irrational number between $\sqrt{5}$ and $\sqrt{13}$.
3. Represent the following real numbers on the number line :
 - (i) $-\sqrt{2}$
 - (ii) $-\sqrt{10}$
 - (iii) $\frac{1}{2}\sqrt{5}$
 - (iv) $\sqrt{10}-2$
4. Find $(\sqrt{2})^3$ upto 4 places of decimals.
5. Give two irrational numbers whose product is rational.
6. Give two irrational numbers whose product is irrational.
7. Is zero a rational or an irrational number ?
8. Give two irrational numbers whose sum is a rational number.
9. Find the value of each of the following up to three places of decimals.
 - (i) $(\sqrt{5})^5$
 - (ii) $(125)^{\frac{1}{3}}$
 - (iii) $(81)^{-\frac{3}{4}}$
 - (iv) $2+(625)^{\frac{3}{4}}$
10. Express each of the following as a product or a quotient of powers in which both a and b (assumed to be positive) appear only once and all exponents are positive.
 - (i) $\frac{5^{-2}a^{-3}b^{-4}}{5^4a^{-5}b^{-6}}$
 - (ii) $\left(\frac{2a^3b^{-1}}{a^{-2}}\right)^5$

11. Which of the following are true ?

(i) $2^3 + 2^4 = 2^7$

(ii) $3^4 \times 3^2 = 3^6$

(iii) $a^5 \times a^{-2} = a^{-10}$

(iv) $(a^{-2} \times a^{-4})^2 = a^{-12}$

12. Determine the value of a in each of the following :

(i) $(\sqrt{5})^4 \times (\sqrt{5})^6 = (\sqrt{5})^{2a}$

(ii) $(\sqrt{2})^3 \div (\sqrt{2})^7 = (\sqrt{2})^{a-2}$

(iii) $(x^{\frac{3}{2}} - 1)(x^{\frac{3}{2}} + 1) = x^{2a+1} - 1$

* (iv) $\sqrt{a\sqrt{3} - \frac{1}{\sqrt{3}}} = \frac{1}{2}$

13. Evaluate each of the following and express the result in the exponential notation :

(i) $[(\sqrt{6})^3 \times \sqrt{6}]^{-\frac{1}{2}}$

(ii) $5^{\frac{2}{5}} \times (5^{\frac{6}{5}} \div 5^{\frac{1}{5}})$

14. If $25 \times (\sqrt{5})^a \times (\sqrt{5})^3 = 5\sqrt{5}$, find a .

15. If $\{36 \times (\sqrt{6})^4\} \div (\sqrt{6})^{\frac{3}{2}} = (\sqrt{6})^{2-a}$, find a .

16. Using the rules learnt so far, simplify the following :

(i) $5 + \frac{1}{3}\sqrt{36}$

(ii) $2 - \frac{1}{4}\sqrt{48}$

17. Simplify the following, assuming that a, b represent positive numbers :

(i) $\frac{\sqrt{a^3b^4}}{\sqrt[3]{b^3a^2}}$

(ii) $\{ \sqrt[5]{12a^5b^3} \div \sqrt[5]{3a^{-2}b^{-1}} \}^{\frac{5}{2}}$

Simplify the following :

18. $(\sqrt{3} x^2 + 10x + 7\sqrt{3}) + (3\sqrt{3} x^2 + 4x - 2\sqrt{3})$

19. $(\sqrt{7} x^3 + 5x - 3\sqrt{3}) + (2\sqrt{7} x^3 + x^2 - 2\sqrt{3}) - (3\sqrt{7} x^3 + x)$

Carry out the indicated operations :

20. $(2x + \sqrt{3}) \times (5x - \frac{1}{\sqrt{3}})$

21. $(1 - \sqrt{2} x) \times (x^2 + 2\sqrt{2} x - 1) + x^2 + \sqrt{2} x + 2$

22. $(x+\sqrt{2}) \times \sqrt{7} x \times (1-x)$

23. $(x^3+3x^2-5) \div (x+2)$

24. $(3x^5-2x^4+x^2-2) \div (x^2+x+1)$

Simplify the following :

25. $\frac{1}{a+b} + \frac{1}{a-b}$

26. $\frac{1}{x} + \frac{1}{x+1} + \frac{1}{x+2}$

27. $\frac{x}{ab} + \frac{x}{bc} + \frac{x}{ac}$

28. $\frac{1}{\sqrt{5}+7x} - \frac{1}{\sqrt{5}-7x}$

29. Carry out the indicated operation :

$$\frac{3x^2+5x}{x^3-3x} - \frac{x^3+3x^2-1}{3x^3-9x}$$

30. If $R = \sqrt{x} + \frac{2}{\sqrt{x}}$, $S = \sqrt{x} - \frac{1}{\sqrt{x}}$, find

(i) $R+2S$

(ii) $R-2S$

31. The area of a rectangular field is $(x^2-7x+12)$ square metres. If one of its sides is $(x-3)$ metres, find the other side.32. Find the sum of the reciprocals of x , $1-x$ and $2+x$.

Find the following products :

33. $(3x+5y)(9x^2-15xy+25y^2)$

34. $(\sqrt{2}x-y)(2x^2+\sqrt{2}xy+y^2)$

Factorize each of the following :

35. $1+8a^3$

36. $125p^3-1$

37. $64k^3+216t^3$

38. $27r^3+d^3$

39. $(1-t)^3+t^3$

40. Simplify :
$$\frac{(302)^3-(300)^3}{(302)^2+(302)(300)+(300)^2}$$

*41. Prove that

$$(x-2y)^3 + (2y-3z)^3 + (3z-x)^3 = 3(x-2y)(2y-3z)(3z-x)$$

UNIT V

Linear Equations and Inequations

In this unit we will study the graphs of linear equations and inequations. We will also learn about simultaneous linear equations in two variables and their solution by graphing and by the method of elimination.

5.1 The Number Plane or the Cartesian Plane

We have learnt in Class VII, how to locate the position of a point in a plane with reference to coordinate axes chosen at our will. It may be recalled that both the coordinates of a point in the first quadrant are positive; in the second quadrant x is negative, y is positive; in the third quadrant both are negative and in the fourth quadrant x is positive, y is negative (See Fig. 5.1). It may also be recalled that the two coordinates have to be written in a particular order, viz., abscissa first and the ordinate second. Thus (3, 4) and (4, 3) are the coordinates of two different points.

We have already learnt that real numbers can be represented by points on a straight line and for this reason the straight line on which the real numbers have been represented is known as the *number line*. Similarly, by introducing the idea of coordinates to locate points in a plane, we have

represented ordered pairs of real numbers by points in the plane. For this reason, the plane on which a coordinate

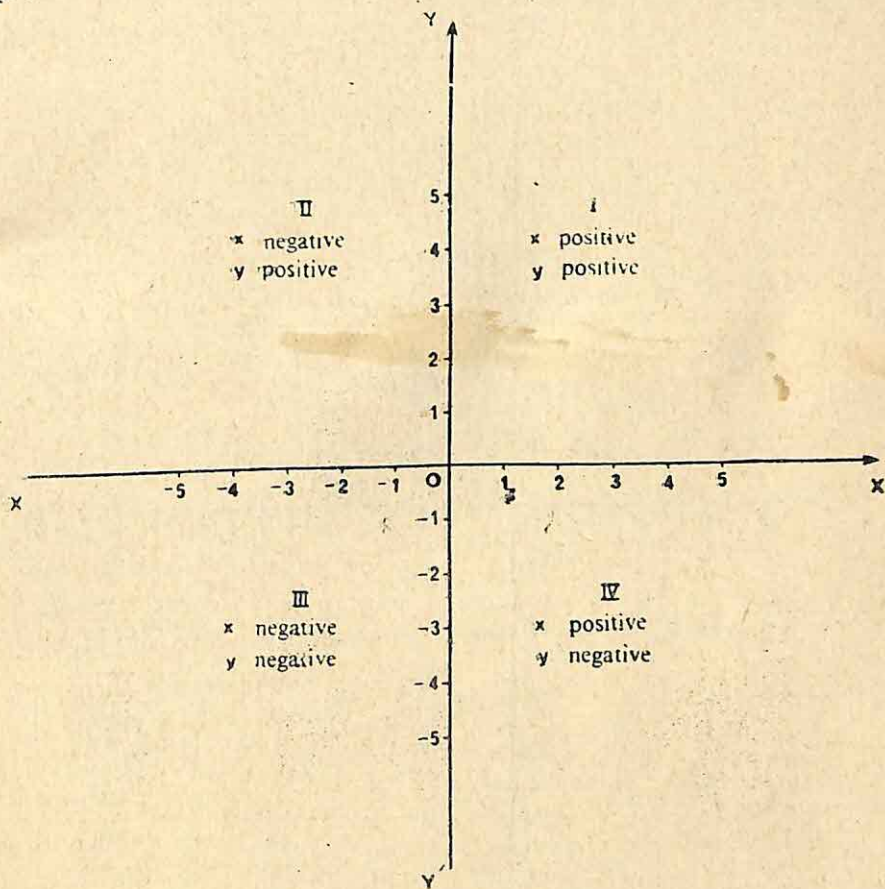


Fig. 5.1

system has been introduced is called a *number plane*. Since the coordinates generally used are Cartesian, the number plane is also called the *Cartesian plane*.

Example 1 : Plot the following points :

$(1, 2)$, $(-2, 3)$, $(-3, -5)$, $(6, -4)$, $(0, 4)$,
 $(2, 0)$, $(0, 0)$, $(-2, -\frac{5}{2})$.

Solution : Let A, B, C, D, E, F, G, H respectively be the

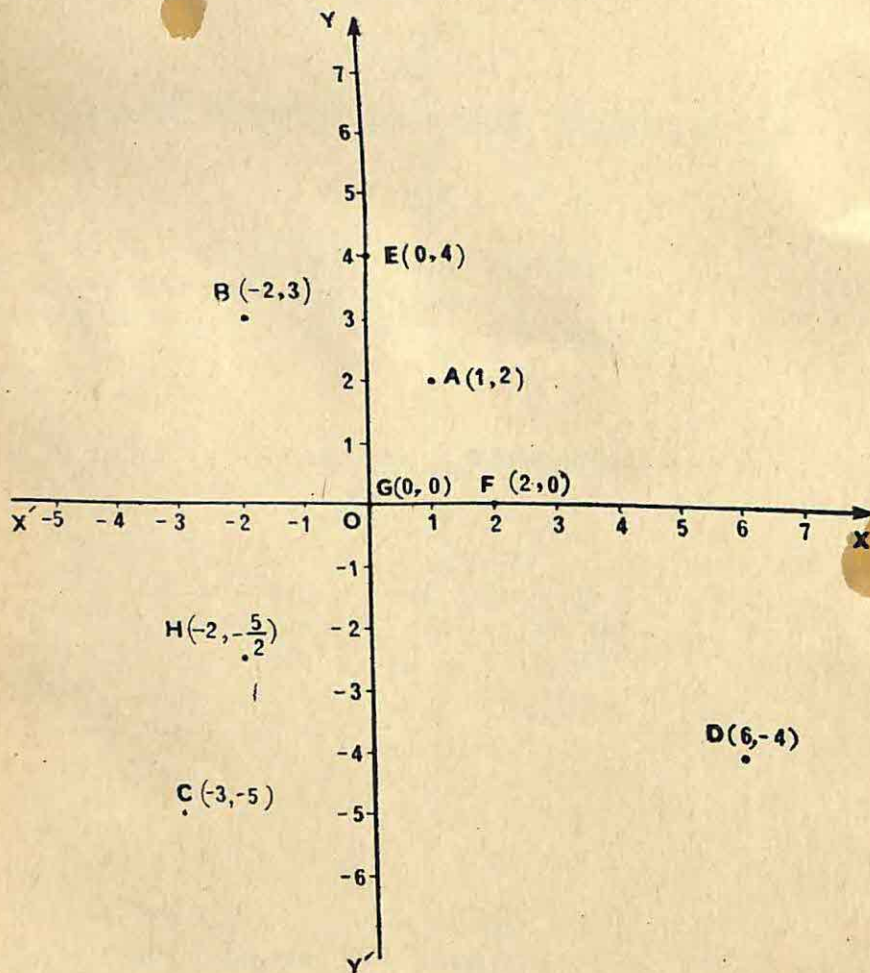


Fig. 5.2

given points. The positions of these points are shown in Fig. 5.2

EXERCISE 5.1

1. Find the coordinates of the points A, B, C, D, E, F, G in the following figure :

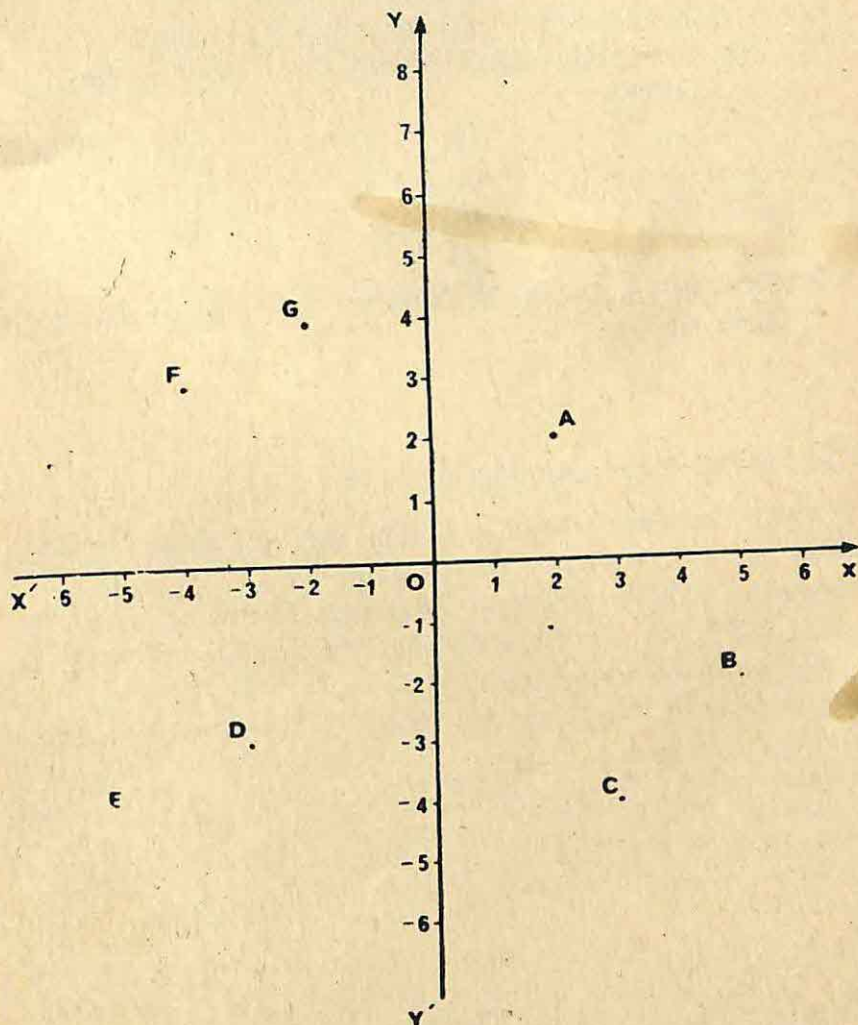


Fig. 5.3

2. Do the ordered pairs (2, 1) and (1, 2) determine the same point of the number plane ?
3. Do the ordered pairs $(-1, 1)$ and $(1, -1)$ determine the same point of the number plane ?
4. Using coordinate axes of your own choice, locate the points determined by the following ordered pairs of numbers :

(i) (2, 3)	(ii) $(-2, -3)$	(iii) $(-2, 3)$
(iv) (3, -2)	(v) (5, 4)	(vi) (4, 5)
(vii) (4, -5)	(viii) $(-4, 5)$	(ix) $(-4, -5)$
(x) (4, 0)	(xi) $(0, -4)$	
5. Give the locations of 5 points of the plane that have abscissa 2.

5.2 Graphical Representation of Data

We give two examples of graphical representation of data in daily life problems.

Example 2 : If train fare per passenger from one station to another is Rs 10/-, the train fare for two passengers shall be Rs 20/-, for three passengers it will be Rs 30/- and so on. This information can be conveniently represented in the form of a table as below :

Number of passengers	1	2	3	4	5	6	7
Train fare in rupees	10	20	30	40	50	60	70

We can also represent this information pictorially by using a Cartesian coordinate system.

If we represent the number of passengers along or parallel to the x-axis and train fare in rupees along or parallel to the y-axis, then the above information may be represented in Fig. 5.4

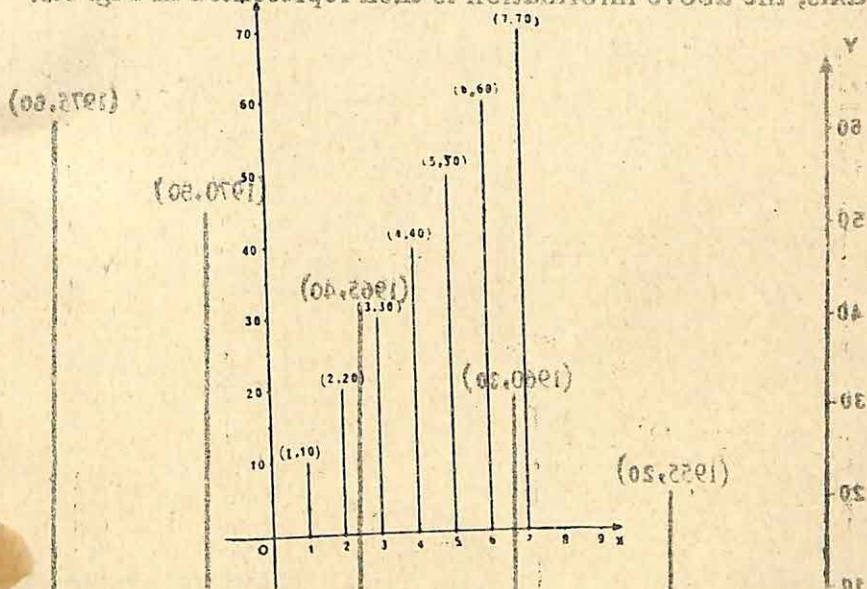


Fig. 5.4

This pictorial representation may be called the graph of train fares against the number of passengers.

Example 3 : The population of a town in 1955 was 20 thousand, in 1960 it was 30 thousand, in 1965 it was 40 thousand while in 1970 and 1975 it was 50 thousand and 60 thousand respectively. We can represent this information in the following table :

Year	1955	1960	1965	1970	1975
Population in thousands	20	30	40	50	60

We again represent this information pictorially using a Cartesian coordinate system.

If we represent the year along or parallel to the x-axis and the population in thousands along or parallel to the y-axis, the above information is then represented in Fig. 5.5.

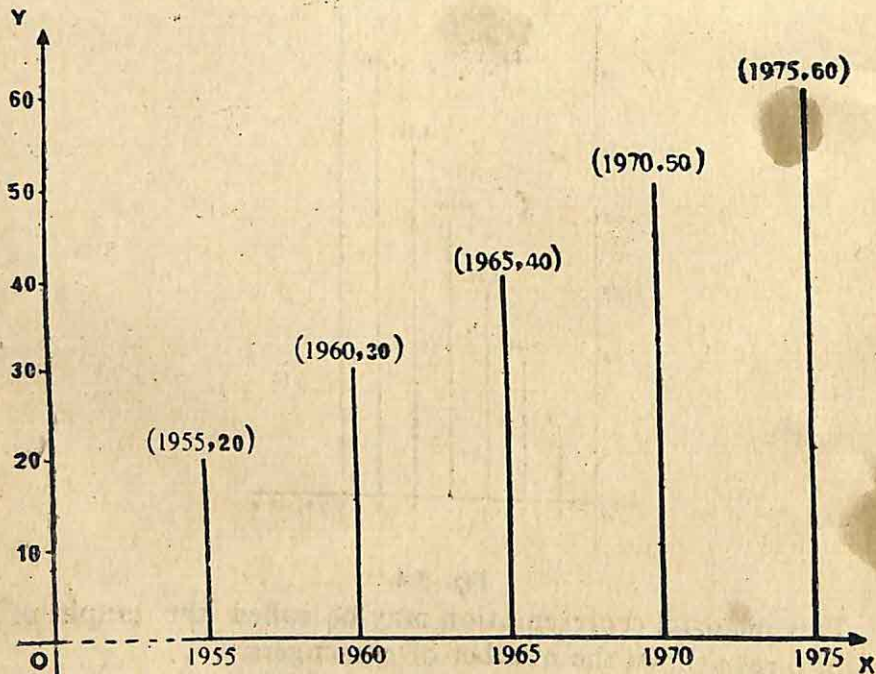


Fig. 5.5

This may be called the graph of the growth of population of the town.

We observe that the graph of any information is a pictorial representation of the information.

In our earlier classes, we have studied linear equations and inequations in one variable. In this and the next section, we shall study their graphs.

5.3 Graphs of Linear Equations in One Variable

We learn about graphs of linear equations in one variable by some examples.

Example 4 : Graph of $x=2$

Let us take some points in the Cartesian plane whose abscissa are all equal to 2, i. e., points for which the x-coordi-

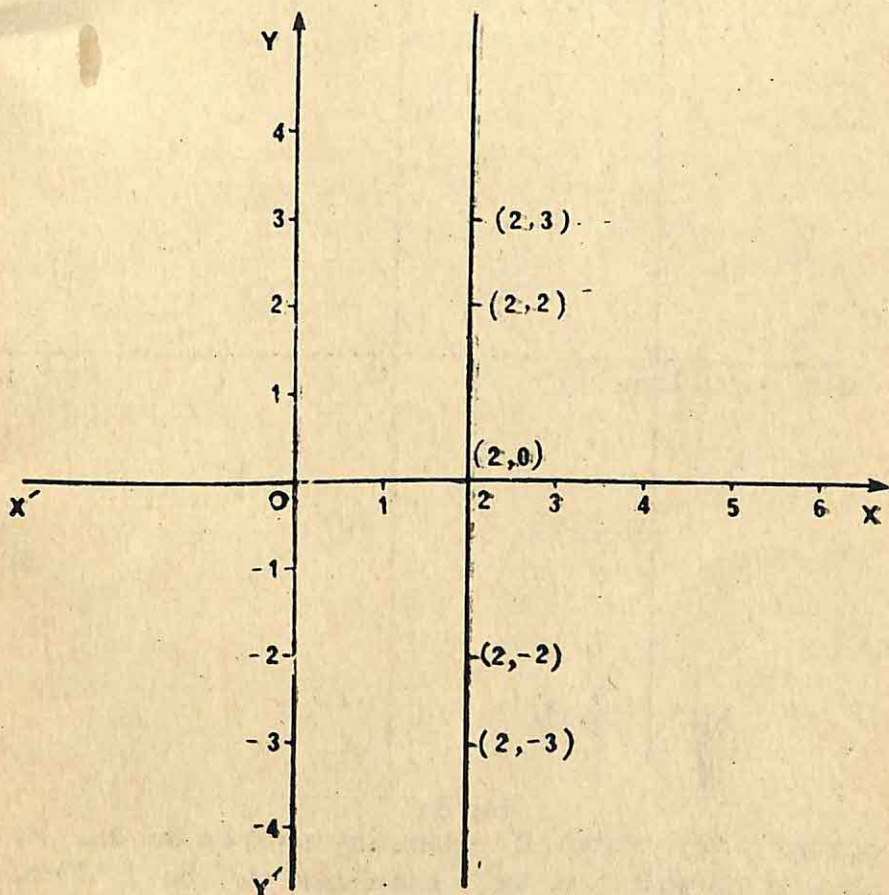


Fig. 5.6

nate is equal to 2. Some examples are, $(2, 0)$, $(2, 2)$, $(2, -2)$, $(2, 3)$ and $(2, -3)$. Let us plot all these points. (See Fig. 5.6) We observe that all these points lie on a straight line which is

Example 4 : Graph of $x = 2$

Let us take some points in the Cartesian plane whose abscissas are all equal to 2, i. e., points for which the x-coordinate is 2.

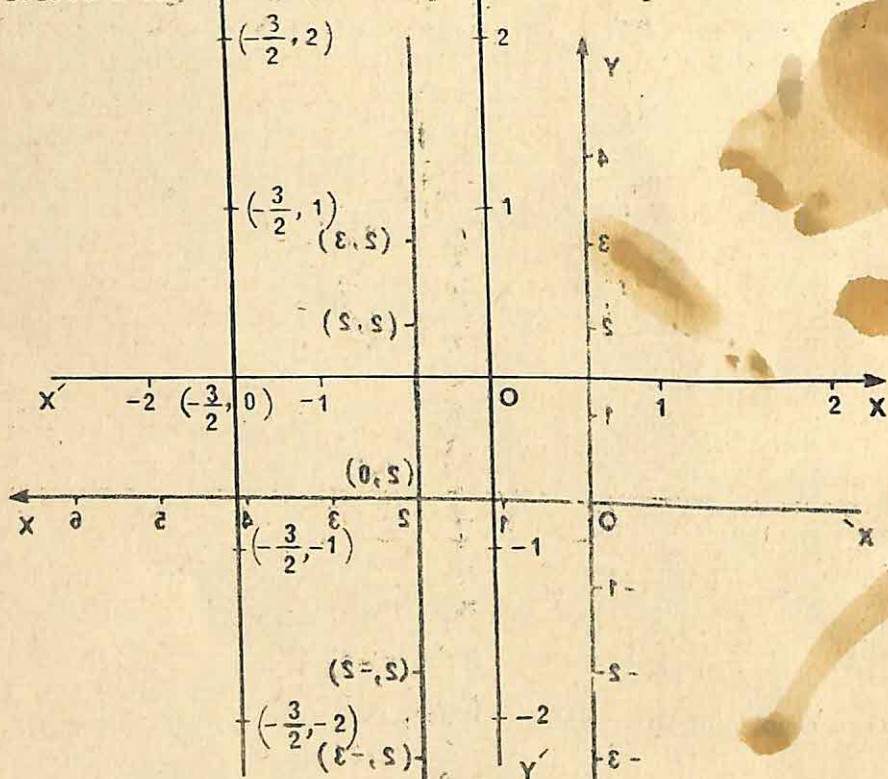


Fig. 5.7

parallel to OY. Again, if we take any point on this line, its abscissa will be 2, i. e., the x-coordinate will be 2. Thus, this line is such that all points whose abscissa x satisfy the

(equation $x=2$, lie on this line. We call this line the graph of the equation $x=2$. We observe that all these points lie on a line.

Example 5: Graph of $2x+3=0$. Again, if we take any point on this line, its ordinate will be $-\frac{2}{3}$, i.e., the y-coordinate will be $-\frac{2}{3}$. This line is such that all points whose x-coordinate is equal to $-\frac{3}{2}$. Some examples are $(-\frac{3}{2}, 0)$, $(-\frac{3}{2}, 1)$, $(-\frac{3}{2}, -1)$, $(-\frac{3}{2}, 2)$, $(-\frac{3}{2}, -2)$. Let us plot all these points. (See Fig. 5.7)

We observe that all these points lie on a straight line which is parallel to the y-axis. Again, if we take any point on this line, its abscissa will be $-\frac{3}{2}$, i.e., the x-coordinate will be $-\frac{3}{2}$. Thus this line is such that all points whose x-coordinate is equal to $-\frac{3}{2}$, lie on this line. We call this line the graph of the equation $x=-\frac{3}{2}$.

Example 6: Graph of $y=\frac{3}{2}$. As before let us take some points in the Cartesian plane. Suppose the yield of rice in a certain village is 50 quintals per hectare. If we denote by x the area of a rice field in hectares and by y the yield in quintals, then

We thus obtain an equation in two variables x and y . In this equation, the larger the area x , larger will be the yield y . Consider another example.

A number y is always 1 less than twice another number x . Then the relation between x and y is

which is again an equation in two variables.

If we take $x=3$, $y=5$, we observe that (1) is a true statement. We say that $x=3$, $y=5$, or that the ordered pair (3, 5) is a solution of the equation (1).

whose ordinates are all equal to $\frac{3}{2}$. Some examples are $(0, \frac{3}{2})$, $(2, \frac{3}{2})$, $(-2, \frac{3}{2})$, $(-4, \frac{3}{2})$, $(4, \frac{3}{2})$, $(-6, \frac{3}{2})$, $(6, \frac{3}{2})$.

Fig. 5.8

Let us plot all these points. We observe that all these points lie on a straight line which is parallel to the x-axis. Again, if we take any point on this line, its ordinate will be $\frac{3}{2}$, i.e., the y-coordinate will be $\frac{3}{2}$. Thus this line is such that all points whose y-coordinate is equal to $\frac{3}{2}$, lie on this line. We call this line the graph of the equation $y=\frac{3}{2}$.

Example 7: Graph of $y=2x$. Let us take some points in the Cartesian plane. Suppose the yield of rice in a certain village is 50 quintals per hectare. If we denote by x the area of a rice field in hectares and by y the yield in quintals, then

We thus obtain an equation in two variables x and y . In this equation, the larger the area x , larger will be the yield y . Consider another example.

A number y is always 1 less than twice another number x . Then the relation between x and y is

which is again an equation in two variables.

If we take $x=3$, $y=5$, we observe that (1) is a true statement. We say that $x=3$, $y=5$, or that the ordered pair (3, 5) is a solution of the equation (1).

$(-2, \frac{5}{2})$, $(-3, \frac{5}{2})$. Let us plot all these points. (See Fig. 5.8) We observe that all these points lie on a straight line which is parallel to OX. Again, if we take any point on this line its ordinate will be $\frac{5}{2}$, i.e., the y coordinate will be $\frac{5}{2}$. Thus, *this line is such that all points whose ordinates y satisfy the equation $y = \frac{5}{2}$, lie on this line.* We call this line the graph of the equation $y = \frac{5}{2}$.

Can you see that we can write $y = \frac{5}{2}$ also as $2y - 5 = 0$?

From the above examples, we note that the graph of an equation in one unknown x or y is a straight line whose abscissa x or ordinates y satisfy the given equation.

5.4 Linear Equations in Two Variables

Recall that an equation in one variable is a statement of equality which contains one unknown. In the same way, an equation in two variables is a statement of equality which contains two unknowns.

Suppose the yield of rice in a certain village is 50 quintals per hectare. If we denote by x the area of a rice field in hectares and by y the yield in quintals, then

$$y = 50x$$

We thus obtain an equation in two variables x and y. In this equation, the larger the area x, larger will be the yield y. Consider another example.

A number y is always 1 less than twice another number x, then the relation between x and y is

$$y = 2x - 1 \tag{1}$$

which is again an equation in two variables.

If we take $x = 3$, $y = 5$, we observe that (1) is a true statement. We say that $x = 3$, $y = 5$, or that the ordered pair (3, 5) is a solution of the equation (1).

Can you find some other solutions of (1)?

Again, $x=4$, $y=7$ make (1) a true statement. Thus the ordered pair (4,7) is also a solution of equation (1).

We observe that in (1), whatever value we give to x , we obtain a corresponding value of y such that the ordered pair so obtained is a solution of equation (1).

Note : The preceding statement is not true for all equations. For example if $2x=3$, we cannot assign any value to x except the value $\frac{3}{2}$. For, no other value of x will make the equation true.

Consider the equation $x=-2$. We may write it as $x+0y=-2$. Thus it can be regarded as an equation in two variables. Similarly, $3y=5$ may be written as $0x+3y=5$, an equation in two variables. We observe that

Every equation in one variable can be regarded as an equation in two variables.

5.4.1 Graphs of Linear Equations in Two Variables

Example 7 : Graph of $2x+3y=6$

Consider the Cartesian plane XOY. We first need to find some ordered pairs which are solutions of the given equation. We would then plot these ordered pairs as points on the plane. By inspection, we observe that (0,2), (3,0), (-3, 4) are some of the ordered pairs that satisfy the given equation. Plot these points on the plane. (See Fig. 5.9). We observe that they all lie on a straight line. We observe further that (9, -4), (-6, 6), (6, -2) are some of the other points that lie on this line. We can verify that they are also solutions of the above equation.

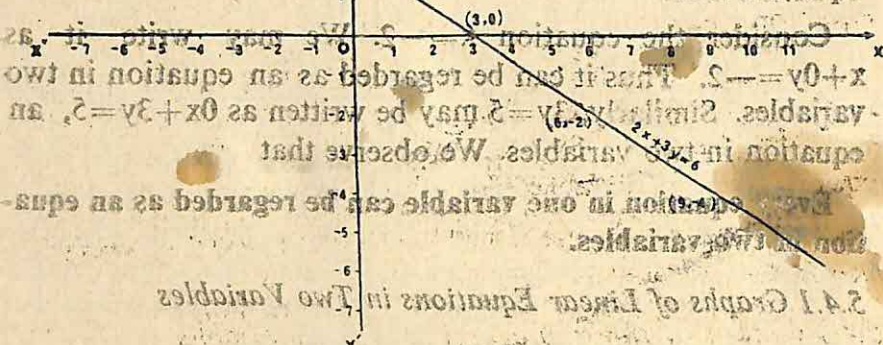
We call this line the graph of the equation $2x+3y=6$.

Let us consider another example.

Again $x=4, y=7$ make (1) a true statement. Thus the ordered pair $(4, 7)$ is also a solution of equation (1).

We observe that in (1), whatever value we give to x , we obtain a corresponding value of y such that the ordered pair so obtained is a solution of equation (1).

Note: The preceding statement is not true for all equations. For example if $2x=5$, we cannot assign any value to x except the value $\frac{5}{2}$. For, no other value of x will make the equation true.



Example 7: Graph of $2x + 3y = 6$
Fig. 5.9

Consider the Cartesian plane XOY. We first need to find some ordered pairs which are solutions of the equation $x + 2y = 3$.

We consider the Cartesian plane XOY. $(1, 1), (3, 0), (-1, 2)$ are some of the solutions of this equation. We plot these points on the Cartesian plane. (See Fig. 5.10). We notice that these points lie on a straight line. Some other points on this line are $(5, -1), (7, -2), (-3, 3), (-5, 4)$. We observe that each of these ordered pairs is also a solution of the given equation.

We call this line the graph of the equation $x + 2y = 3$.

We observe that the graph of a linear equation in one or two variables is always a straight line. Also recall that a straight line is located as soon as we know two points on the

5.2 Graphs of Linear Functions in One Variable

Recall that a linear inequality in one variable is a statement of inequality which contains one variable. We now learn about graphs of such linear inequalities.

Example 2 : Graph of the inequality $x < 3$.

Solution : We first draw a straight line parallel to the y-axis such that the abscissa of every point on this line is 3. This line divides the plane into two regions, I and II, say. (See Fig. 5.11) Let us consider any point in region I. One such convenient point is (1, 1). Obviously (1, 1) satisfies the inequality $x < 3$. Similarly every other point in region I has abscissa < 3 and, so, satisfies the inequality $x < 3$.

Fig. 5.10

line. Thus, in order to obtain the graph of a linear equation, it is sufficient to obtain two solutions of the equation. Then, the straight line joining the two points corresponding to these two solutions is the graph of the equation.

EXERCISE 5.2

- Draw the graphs of the following equations :
(i) $2x = -3$ (ii) $y = -4$ (iii) $2y = -5$ (iv) $3x = 4$.
- Draw the graph of the equation $y = -x$. From the graph, read the value of x when $y = 2$, and the value of y when $x = -3$.
- Draw the graph of $x + y = -3$. From the graph, read the value of y when $x = -\frac{3}{2}$, and the value of x when $y = \frac{5}{2}$.
- Draw the graphs of each of the following :
(i) $3x + 4y = 6$; (ii) $y - 3x = 4$
(iii) $y = -2x + 1$; (iv) $x - y + 3 = 0$.

5.5 Graphs of Linear Inequations in One Variable

Recall that a linear inequation in one variable is a statement of inequality which contains one unknown. We now learn about graphs of such linear inequations.

Example 9 : Graph of the inequation $x \leq 3$.

Solution : We first draw the graph of the equation $x=3$. This is a straight line parallel to the y-axis such that the abscissa of every point on this line is 3. This line divides the plane into two regions, I and II, say. (See Fig. 5.11) Let us consider any point in region I. One such convenient point is (1, 1). Obviously (1, 1) satisfies the inequation $x \leq 3$. Similarly every other point in region I has abscissa < 3 and, so, satisfies the inequation $x \leq 3$.

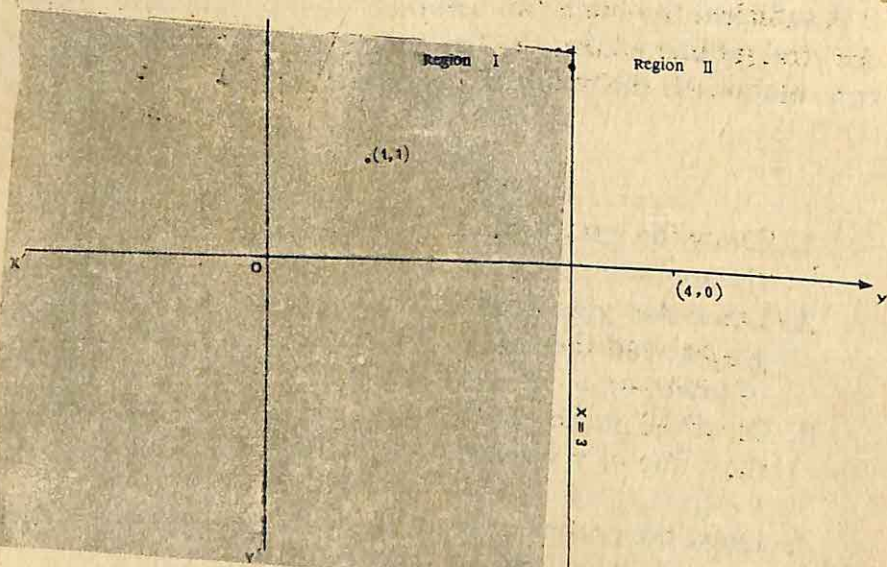


Fig. 5.11

Next we consider any point in region II. $(4, 0)$ is one such convenient point. The ordered pair $(4, 0)$ does not satisfy the inequation $x \leq 3$. Also, clearly, any other point in region II has abscissa > 3 and therefore does not satisfy the inequation $x \leq 3$. Every point on the line $x=3$ which separates the regions I and II has its abscissa $x=3$ and thus satisfies the inequation $x \leq 3$. Since the abscissa of every point in region I or on the line $x=3$ satisfies the inequation $x \leq 3$, **Region I together with the line $x=3$ is called the graph of the inequation $x \leq 3$.**

Example 10 : Graph of the inequation $y > -2$

Solution : We first draw the graph of the equation $y = -2$.

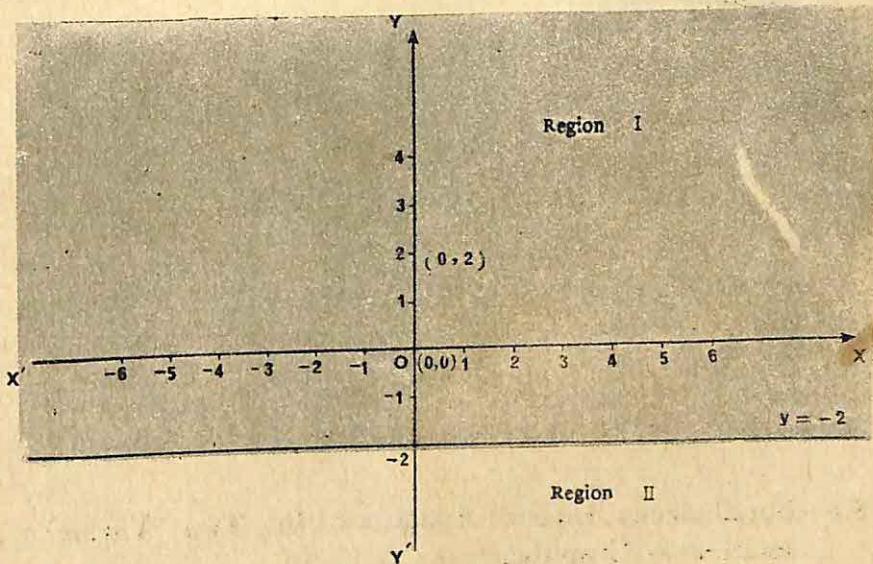


Fig. 5.12

It is a straight line parallel to the x -axis and passes through the point $(0, -2)$. This line divides the plane XOY into two regions, say I and II. (See Fig. 5.12).

Consider (any) point in region I. $(0, 0)$ is one such convenient point. The ordered pair $(0, 0)$ satisfies the inequality $y \geq -2$. Also, every other point in region I has an ordinate ≥ -2 . The corresponding ordered pair satisfies the inequality $y \geq -2$. Every point in region II has ordinate < -2 . The corresponding ordered pair does not satisfy the inequality $y \geq -2$. Also, every point on the line which separates the regions I and II, satisfies $y = -2$, and so satisfies $y \geq -2$. We call region I together with the line $y = -2$ the graph of the inequality $y \geq -2$.

To sum up, we say that the graph of an inequality is the region of the plane, every point of which satisfies the given inequality.

EXERCISE 5.3

Draw the graph of each of the following inequations :

1. $x > -3$

3. $2y + 3 \geq 0$

5. $x < 0$

7. $y > 0$

(5, 0)

2. $y < -2$

4. $3x + 6 \leq 0$

6. $x \leq 0$

5.6 Simultaneous Linear Equations in Two Variables : Solution by Graphing

Example 11 : A question paper of a class, divided into two parts, A and B. All questions in Part A carry equal marks and all questions of part B also carry equal marks. Ram correctly attempts 2 questions from part A and 3 questions

from part B. He secures 40 marks. Rahim correctly solves 3 questions from part A and 2 questions from part B. He secures 35 marks. Determine the marks allotted to each question of part A and that of part B.

(Solution : Suppose each question of part A carries x marks. Similarly, for the marks secured by Rahim, we get the

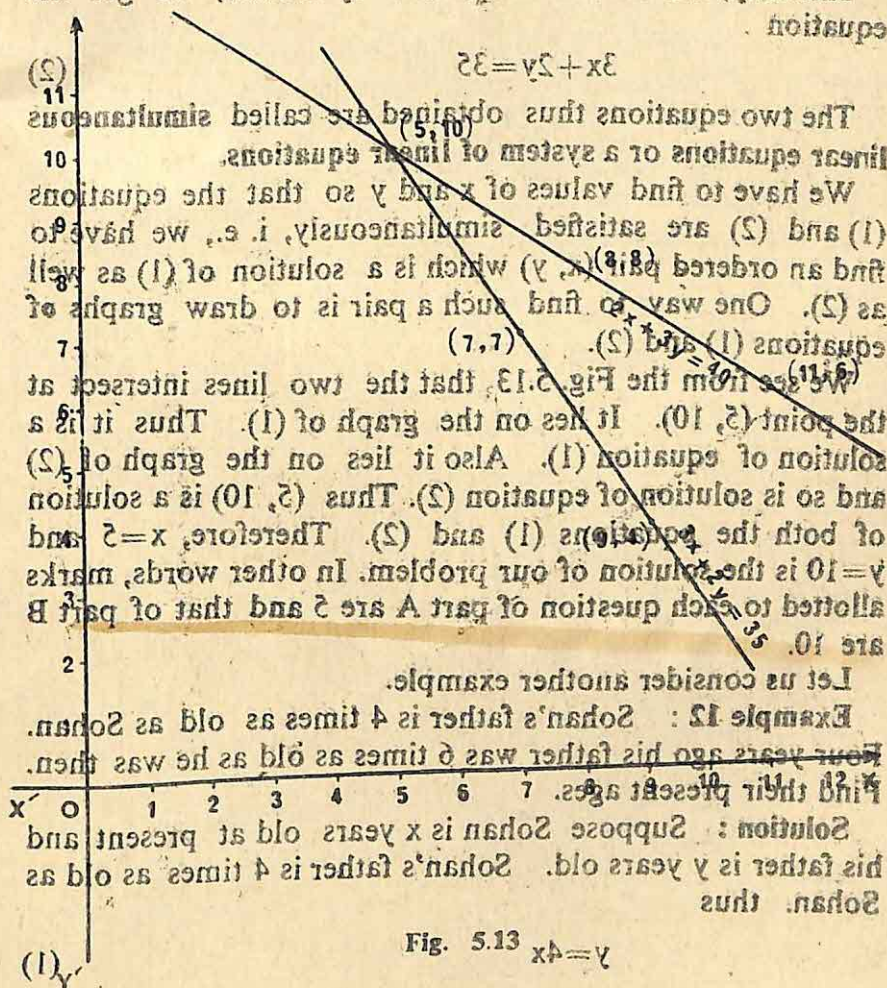


Fig. 5.13 $y = 4x$

Let us consider another example.
Example 12 : Sohan's father is 4 times as old as Sohan. Four years ago his father was 6 times as old as he was then. Find their present ages.
Solution : Suppose Sohan is x years old at present and his father is y years old. Sohan's father is 4 times as old as Sohan. thus

and each question of part B carries y marks. Ram solves correctly 2 questions from part A and 3 questions from part B. Therefore, his score will be $2x + 3y$. But he secures 40 marks. We then get the equation

$$2x + 3y = 40 \quad (1)$$

Similarly, for the marks secured by Rahim, we get the equation

$$3x + 2y = 35 \quad (2)$$

The two equations thus obtained are called **simultaneous linear equations** or a **system of linear equations**.

We have to find values of x and y so that the equations (1) and (2) are satisfied simultaneously, i. e., we have to find an ordered pair (x, y) which is a solution of (1) as well as (2). One way to find such a pair is to draw graphs of equations (1) and (2).

We see from the Fig. 5.13 that the two lines intersect at the point $(5, 10)$. It lies on the graph of (1). Thus it is a solution of equation (1). Also it lies on the graph of (2) and so is solution of equation (2). Thus $(5, 10)$ is a solution of both the equations (1) and (2). Therefore, $x=5$ and $y=10$ is the solution of our problem. In other words, marks allotted to each question of part A are 5 and that of part B are 10.

Let us consider another example.

Example 12 : Sohan's father is 4 times as old as Sohan. Four years ago his father was 6 times as old as he was then. Find their present ages.

Solution : Suppose Sohan is x years old at present and his father is y years old. Sohan's father is 4 times as old as Sohan. thus

$$y = 4x \quad (1)$$

Their ages were $x-4$ and $y-4$ respectively four years ago. Therefore from the second given condition

$$y-4=6(x-4)$$

or, $y=6x-20$ (2)

We need to find the values of x and y that satisfy both (1) and (2).

We draw the graphs of equations (1) and (2). These are two straight lines. (See Fig. 5.14) From the figure we see that these lines intersect at the point $(10, 40)$. Thus $x=10$, $y=40$ satisfy both the equations and is therefore the solution of our problem. In other words, the present ages of Sohan and his father are 10 years and 40 years respectively.

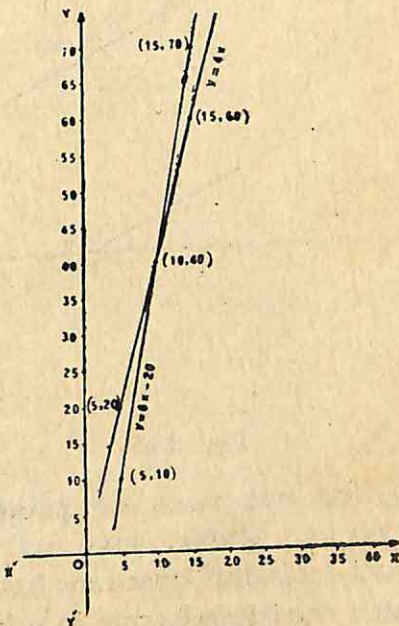


Fig. 5.14

annual premia payable on different sums assured for different periods of time.

(2) *Conversion Tables*: We can convert from one system of units to another system of units for the measurement of the same quantity with these tables. For example we can use these tables for conversion of the measurement of lengths from the older system to the metric system, and vice versa.

EXERCISE 6.2

Using the Table 6.2, calculate the yearly interest on

1. Rs 50/- at the rate of 4% per annum.
2. Rs 90/- at the rate of $4\frac{1}{2}\%$ per annum.
3. Rs 400/- at the rate of 5% per annum.
4. Rs 500/- at the rate of $4\frac{1}{2}\%$ per annum.
5. Calculate interest on Rs 70/- at the rate of 5% per annum for 11 months.
6. Calculate interest on Rs 300/- at the rate of 4% per annum for 3 months.

UNIT VII

Sets

In this unit, we learn about sets, set notation and some operations on sets. We also learn to represent sets pictorially.

7.1 Sets

Sets and set language have come to occupy an important position in mathematics in recent times. They help us to express many of our ideas and concepts in a clear and concise manner. All of us have an intuitive notion of what we understand by a collection, a group, an aggregate, or a set of things in everyday language. Let us now consider a few examples of collections to discover what characteristics we expect of the mathematical notion of a set.

1. All the students in your class.
2. A family consisting of a man, his wife, their two sons and a daughter.
3. All the buffaloes in your village.
4. All the letters of the English alphabet.
5. The natural numbers greater than two but less than ten.
6. The even natural numbers.

We observe that to find a solution of a system of two simultaneous linear equations, we draw the graphs of the two equations. The point of intersection, if any, of the two graphs is a solution of the system of equations.

(5.6.1. Does a system of two simultaneous linear equations always have a solution? Not necessarily, as is evident from the following example.

Example 13 : Consider the system of equations:

$$x + 2y = 4 \quad \text{and} \quad x + 2y = 8$$

The graphs of these equations are shown in Fig. 5.15.

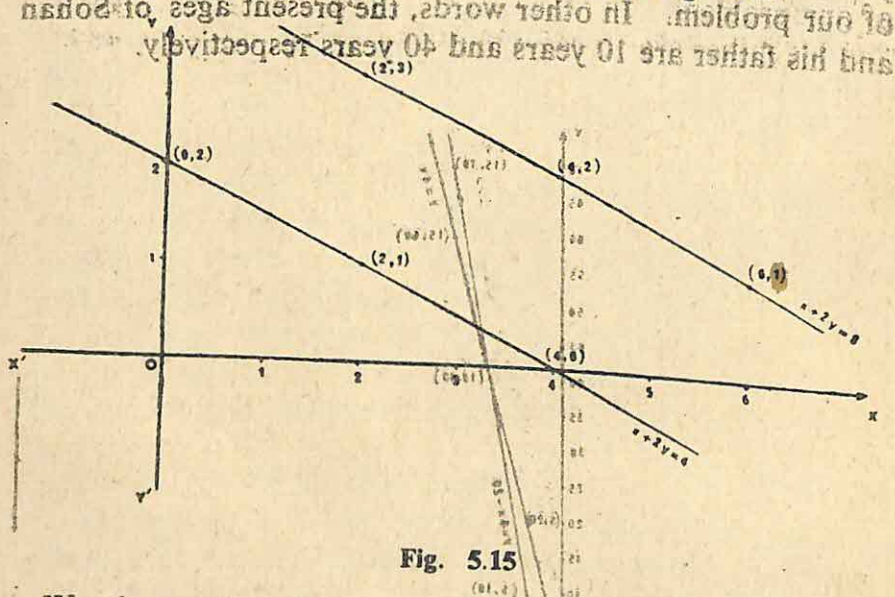


Fig. 5.15

We observe that the two lines are parallel and do not intersect. Thus the two graphs have no common point. Therefore, the given system of equations has no solution.

A system of linear equations having no solution is called a system of inconsistent equations.

Observe that $x+y=3$, $x+y=8$ is another system of inconsistent equations. Verify.

EXERCISE 5.4

Solve the following equations graphically.

1. $x+3y=12$

$3x+y=12$

3. $x+2y-3=0$

$-x-2y+6=0$

5. $2x+3y=5$

$3x+2y=10$

7. $2x-4y=3$

$3x-6y=1$

2. $x-2y=16$

$2x+y=2$

4. $x+y-6=0$

$2x+3y=12$

6. $-x+2y+10=0$

$2x-y-8=0$

8. $x-y=3$

$x+y=5$

5.7 Solving two Simultaneous Linear Equations in two Variables

5.7.1 Method of Elimination by Addition and Subtraction

Example 14 : Solve the system of equations

$$3x-2y=19$$

$$x+y=23$$

Solution : Recall that equality sign in an equation is not affected if both sides of an equation are multiplied by the same number except zero. We multiply both sides of the second equation by 2. The system of equations becomes

$$3x-2y=19$$

$$2x+2y=46$$

We find that now coefficient of y in both the equations is

numerically the same. If we add these equations, the terms involving y cancel out, and we get

$$5x = 65$$

Dividing both sides by 5, we get $x = 13$.

Substitute $x=13$ in the second equation. We get

$$13+y=23$$

$$y=-13+23=10$$

Hence $x=13, y=10$ is a solution of the given system of equations.

Let us verify our solution : When we substitute these values of x and y in the given equations, we get

$$3x-2y=3 \times 13-2 \times 10=39-20=19$$

and

$$x+y=13+10=23$$

Thus, the equations are satisfied and our solution is correct.

Example 15 : Solve the system of equations

$$3x-2y=-10 \quad (1)$$

$$4x+y=49 \quad (2)$$

Solution : Multiply both sides of the second equation by 2. We get

$$8x+2y=98 \quad (3)$$

Observe that coefficients of y in equations (1) and (3) are numerically equal. We add (1) and (3). We get

$$11x=88 \text{ or } x=8$$

Put this value of x in (2). We get

$$32+y=49 \text{ or } y=17$$

Hence $(8, 17)$ is a solution of the given system. Verify.

Example 16 : Solve the system of equations

$$2x + 3y = 10 \quad (1)$$

$$3x + 2y = 5 \quad (2)$$

Solution : Multiply both sides of equation (1) by 3 and those of (2) by 2. We get

$$6x + 9y = 30 \quad (3)$$

$$6x + 4y = 10 \quad (4)$$

Coefficients of x in both the equations are now equal. Subtract (4) from (3). We get

$$5y = 20 \text{ or } y = 4.$$

Put $y = 4$ in (1). We get

$$2x + 12 = 10 \text{ or } 2x = -2, \text{ i.e., } x = -1$$

Hence $(-1, 4)$ is a solution of the given system of equations. Verify.

We observe that the method of solution by elimination using addition or subtraction consists of the following steps :

Step I We compare the coefficients of the variables in the two equations.

Step II We multiply (or divide) the equations by suitable number so that coefficients of one of the variables are numerically equal in the two equations.

Step III We add the two equations or subtract one from the other so that the variable with equal numerical coefficient gets eliminated.

Step IV We solve the resulting equation in the remaining variable and find the value of this variable.

Step V We substitute the value of the variable obtained in step IV in one of the original equations. This

gives an equation in the other variable. Solve this single equation and we get the value of the other variable also.

Step VI Check that the given equations are satisfied by the values of the unknowns obtained above.

EXERCISE 5.5

Solve the following systems of equations by the method of elimination by addition and subtraction.

1. $x - y = -1$
 $10x - 8y + 7 = 0$

3. $6x - 2y = 1$
 $3x + 10y = 6$

5. $2x + 3y = 5$
 $3x + 2y = 10$

7. $x - y = 3$
 $3x + 4y = 10$

2. $2x + 3y + 5 = 0$
 $3x - 2y - 12 = 0$

4. $5x + 9y = 89$
 $2x - 17y = 15$

6. $4x - 3y = 7$
 $3x - 4y = 14$

5.7.2 Method of Elimination by Substitution

Example 17 : Solve the system of equations

$$\begin{aligned}x - y &= 3 & (1) \\3x + 4y &= 9 & (2)\end{aligned}$$

Solution : From equation (1) we get
 $-y = 3 - x$ or $y = x - 3$.

Put this value of y in (2), we get

$$3x + 4(x - 3) = 9$$

or, $3x+4x-12=9$ or $7x=21$, i.e., $x=3$

Put $x=3$ in (1). We get $3-y=3$ or $y=0$

Hence $(3, 0)$ is the solution of the system of equations (1) and (2)

Can you verify this ?

We observe that the method of elimination by substitution consists of the following steps :

Step I From one of the equations, we express one unknown in terms of the other.

Step II Substitute this value of the unknown in the second equation. We get an equation in a single variable. Solve this linear equation in one variable. This gives the value of one unknown.

Step III Substitute the value of the unknown obtained in step II in one of the original or derived equations and then solve for the other unknown.

Note that the two methods are essentially the same. In each we eliminate one of the two variables to get a simple equation in the other variable. Let us consider another example.

Example 18 : Solve the system of equations :

$$x-2y=6 \quad (1)$$

$$2x+y=2 \quad (2)$$

Solution : From (2), we get

$$y=2-2x \quad (3)$$

Substitute this value of y in (1). We get

$$x-2(2-2x)=6$$

or, $x-4+4x=6$ or $5x=10$, i.e., $x=2$.

Put $x=2$ in (3) We get

$$y = 2 - 4 = -2.$$

Hence $(2, -2)$ is the solution of the given system. Verify.

Example 19 : Solve the system of equations :

$$2x + y = 3 \quad (1)$$

$$2y - 3x = 6 \quad (2)$$

Solution : From (1), we get

$$y = 3 - 2x. \quad (3)$$

Put this value of y in (2). Then

$$2(3 - 2x) - 3x = 6$$

or, $6 - 4x - 3x = 6$ or $7x = 0$, i.e., $x = 0$.

Put $x = 0$ in (3). we get $y = 3$.

Hence $(0, 3)$ is the solution of the given system of equations. Verify.

EXERCISE 5.6

Solve the following system of equations by using the method of elimination by substitution.

1. $x + y = 60$
 $2x + 5y = 165$

2. $9x + 4y = 19$
 $-9x + 30y = 270$

3. $2x + 7y = 13$
 $2x + 2y = 5$

4. $4x + 3y = 7$
 $10x - 3y = 7$

5. $2x + 3y - 8 = 0$
 $5x - 8y + 11 = 0$

6. $2x + 5y = 34$
 $x + 3y = 20$

7. $3x - 5y = 7$
 $2x + 3y = 12$

5.8 Linear Inequalities in two Variables

Two brothers, Ram and Shyam, go to see a village fair and their father gives them Rs 10/- as pocket money to be spent by them at the fair. They are allowed to spend the money as they like. Suppose x is the amount spent by Ram while y is the amount spent by Shyam. Since the two together cannot spend more than Rs 10/-, we have $x+y \leq 10$. This is an inequality in two variables. In general,

A statement of inequality which contains two unknown quantities is called an inequality in two variables. We will study such inequalities and their graphs in higher classes.

Key Concepts

Graphs of linear equations in one and two variables	Simultaneous equations :
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Linear equations in two variables	Solution by graphing
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Graphs of linear inequalities in one variable	Method of elimination
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UNIT VI

Use of Tables

In this unit we will learn about the use of tables to find squares, square roots, cubes, cube roots etc. of natural numbers. We will learn also about the use of tables in finding interest.

6.1 Introduction

There are many numerical computations in mathematics which have to be made repeatedly in various types of problems. For example, we have to compute often the squares, cubes, square roots, cube roots, etc. of natural numbers. Instead of making such computations every time that they are needed, we can make such calculations once for all and write them down in the form of tables from which the required values can be just read. This would greatly save our labour. In fact, such calculations have been made for many types of mathematical expressions and are available in printed form. One set of such tables is the well known 'Barlow's Tables of Squares, Cubes, etc.' of all natural numbers from 1 to 12,500.

In this unit we learn how to use mathematical tables, taking our examples from tables of squares, cubes, etc. and interest tables.

No. n	Square n^2	Cube n^3	Square root $\sqrt{n}$	Cube root $\sqrt[3]{n}$
1	1	1	1.000	1.000
2	4	8	1.414	1.259
3	9	27	1.732	1.442
4	16	64	2.000	1.587
5	25	125	2.236	1.709
6	36	216	2.449	1.817
7	49	343	2.645	1.912
8	64	512	2.828	2.000
9	81	729	3.000	2.080
10	100	1000	3.162	2.154
11	121	1331	3.316	2.223
12	144	1728	3.464	2.289
13	169	2197	3.605	2.351
14	196	2744	3.741	2.410
15	225	3375	3.872	2.466
16	256	4096	4.000	2.519
17	289	4913	4.123	2.571
18	324	5832	4.242	2.620
19	361	6859	4.358	2.668
20	400	8000	4.472	2.714
21	441	9261	4.582	2.758
22	484	10648	4.690	2.802
23	529	12167	4.795	2.843
24	576	13824	4.898	2.884
25	625	15625	5.000	2.924

Table 6.1 : Table of values of squares, cubes, square roots, and cube roots of all natural numbers from 1 to 25. The values in the columns of square roots and cube roots are given up to three places of decimals.

6.2 Use of Tables in Computing Squares, Cubes, Square Roots and Cube Roots of Natural Numbers

Let us look at Table 6.1. This is an extract from Barlow's Tables for the values $n=1$ to $n=25$ of the natural numbers.

Barlow's Tables give also the values of $\frac{1}{n}$ and $\frac{1}{\sqrt{n}}$ and some other constants. For any extended computation involving the arithmetical operations on natural numbers, we can use the main Barlow Tables or other similar tables.

We observe that the first column gives the natural numbers from 1 to 25. The second, third, fourth and fifth columns give, opposite these numbers, their squares, cubes, square roots, and cube roots respectively. We now learn the use of this table from some examples.

Example 1 : Find $(13)^3$.

Solution : We first locate 13 in the first column. Then move along the horizontal line containing 13 and locate the entry in this row in the column of cubes. We find that the entry there is 2197. Thus,

$$(13)^3 = 2197$$

Example 2 : Find $\sqrt{5}$.

Solution : We first locate the number 5 in the first column. We then go along the horizontal line which contains 5 and locate the entry in this row in the column of square roots. We find that the entry there is 2.236. Thus,

$$\sqrt{5} = 2.236$$

Example 3 : Find $\sqrt[3]{16}$.

Solution : We first locate the number 16 in the first column. We then go along the horizontal line which contains 16 and locate the entry in this row in the column of cube roots. We find that this entry is 2.519. Thus,

$$\sqrt[3]{16} = 2.519$$

EXERCISE 6.1

Using the Table 6.1, find

- | | |
|---------------------------|---------------------------|
| 1. the square of 19. | 2. the square of 23. |
| 3. the cube of 11. | 4. the cube of 17. |
| 5. the cube of 24. | 6. the square root of 7. |
| 7. the square root of 18. | 8. the square root of 21. |
| 9. the cube root of 12. | 10. the cube root of 15. |

6.3 Use of Interest Tables

We know that when we deposit money in a Savings Bank account in a bank or a post office, the bank or the post office pays us, in addition to the amount deposited, interest on this amount. Now the amount of interest depends upon the amount deposited, the rate of interest and the time for which the amount remains deposited. This interest is usually added to the account once in a year at the same time for all the depositors or when a depositor withdraws all his deposit from the bank. In any bank or post office, a very large number of interest calculations have to be made every year. These calculations are made more tedious by frequent deposits and withdrawals during the year. It would be very laborious if such calculations have to be made afresh in the case of each account. To avoid the greater part of such labour, interest calculations for one year on different amounts at various rates of interest, say from 1 per cent to 10 per cent at intervals of $\frac{1}{2}$ per cent, are made once for all and printed as interest tables. With the help of these tables, the bank officials can calculate the interest due to each depositor very quickly.

Let us look at Table 6.2.

Rate Amount		4%		4½%		5%	
Rs		Rs	P	Rs	P	Rs	P
1	...	...	04	...	05	...	05
2	...	...	08	...	09	...	10
3	...	...	12	...	14	...	15
4	...	...	16	...	18	...	20
5	...	...	20	...	23	...	25
6	...	...	24	...	27	...	30
7	...	...	28	...	32	...	35
8	...	...	32	...	36	...	40
9	...	...	36	...	41	...	45
10	...	...	40	...	45	...	50
20	...	...	80	...	90	1	00
30	1	20	...	1	35	1	50
40	1	60	...	1	80	2	00
50	2	00	...	2	25	2	50
60	2	40	...	2	70	3	00
70	2	80	...	3	15	3	50
80	3	20	...	3	60	4	00
90	3	60	...	4	05	4	50
100	4	00	...	4	50	5	00
200	8	00	...	9	00	10	00
300	12	00	...	13	50	15	00
400	16	00	...	18	00	20	00
500	20	00	...	22	50	25	00
600	24	00	...	27	00	30	00
700	28	00	...	31	50	35	00
800	32	00	...	36	00	40	00
900	36	00	...	40	50	45	00
1000	40	00	...	45	00	50	00

Table 6.2 (Table giving interest per annum).

Observe that the first column indicates the amount in rupees, the second column gives interest at the rate of 4% per annum, the third column gives the interest at the rate of 4½% while the fourth column gives the interest at the rate of 5%.

Example 4 : Calculate the yearly interest on Rs 60/- at the rate of 5% per annum.

Solution : We locate 60 in the first column. In the horizontal line containing 60, we then locate the entry in the column of 5%. This entry gives Rs 3. Thus the yearly interest on Rs 60 at the rate of 5% per annum is Rs 3.

Example 5 : Calculate the yearly interest on Rs 300/- at the rate of $4\frac{1}{2}\%$ per annum.

Solution : We locate 300 in the first column. In the horizontal line containing 300, we then locate the entry in the column of $4\frac{1}{2}\%$. This entry gives Rs 13 and paise 50. Thus the yearly interest on Rs 300 at the rate of $4\frac{1}{2}\%$ per annum is Rs 13 and paise 50.

Example 6 : Calculate the interest on Rs 80/- at the rate of 5% per annum for 4 months.

Solution : We locate 80 in the first column. In the horizontal line containing 80, we then locate the entry in the column of 5%. This entry gives Rs 4. So the interest for one year on Rs 80 is 400 paise. Therefore interest for 4 months on Rs 80 is $(\frac{400}{12} \times 4)$ paise = $\frac{400}{3}$ paise, i.e., nearly 133 paise. Thus the interest on Rs 80/- at the rate of 5% per annum for 4 months is Re 1 and paise 33.

Remark : Table 6.2 provides interest at the rate of 4%, $4\frac{1}{2}\%$ and 5% per annum. This table does not give us the interest on Rs 150 at the rate of 10% per annum directly, as it is incomplete. But we have tables which give us the interest at other rates on different amounts

Many other types of tables are available. We mention below two of them.

(1) *Insurance Premium Tables*. These give the amounts of

7. Five most interesting books in a certain school library.
8. The letters in the word SCHOOL.
9. The letters in the word Committee.
10. Three strongest men of India.

Let us look at Example 1. Do we know for definite what belongs to this collection and what does not? Yes, we do. For anything to belong to this collection, it has first to be a student, and next a student in the same class as you are. Are you yourself a part of this collection? Yes, you are. *So we know who belongs to the collection and who does not.* Also we note that all the members in this collection are *different or distinct*.

Can we say the same about all the other collections? Let us check. In Example 2, we know for definite who belongs to the family and who does not. Also all the members are distinct. Evidently, the same can be said about the collections in Examples 3-6. But what about Example 7? Can we say definitely which are the most interesting books in the school library? No, because what may appear interesting to one person may not appear so to another. Thus the collection in Example 7 is not well determined or well defined. Evidently, we cannot do much with such a collection as we do not know for certain what belongs to it and what does not. Example 10 is similar. Why? Such collections are not regarded as sets in mathematics.

Next, let us look at Example 8. The letters in the word SCHOOL are S, C, H, O, O, L, six in number. Although given any letter of the alphabet, we can say immediately whether it belongs to this collection or not. However, the letter O occurs twice. When any object appears in a collection

more than once, it is found convenient to count it only once, i. e., as only one object, and not as many times as it occurs. With this understanding the objects in such a collection also become distinct. Thus, the collection of letters in the word SCHOOL would consist simply of the five letters S, C, H, O, L.

Again, in example 9, the collection of letters in the word *Committee* consists simply of six distinct letters c, o, m, i, t, e. It may also be observed that in this manner, we are considering the collections, such as c, o, m, m, i, t, t, e, e. and c, o, m, i, t, e as the same.

From the above considerations, we say that :

By a set we will understand a well defined collection of distinct objects. Thus 1-6 and 8, 9 above are examples of sets.

The objects belonging to a set are called its *members* or *elements*.

Although the elements of a set are generally objects of the same kind, it is not necessary. They may be of different kinds. For example, we may have a set consisting of a horse, a cart and a man. Also, we may have sets as elements of sets. We shall generally be concerned with sets whose elements are numbers. Another important category of sets that we deal with are sets of points. We give a few more examples of sets.

11. All citizens of India.
12. All the points on a given straight line.
13. All the points lying on a given circle.
14. All the integers.
15. All the real numbers.
16. The set of letters in the word 'success'. Notice that by the convention explained for Examples 8, 9,

this set consists only of the letters s, u, c, e, four in all.

EXERCISE 7.1

Which of the following are sets ? For those that are not, explain why.

1. The collection of all mango trees in a certain garden.
2. The collection of five most popular teachers in your school.
3. The collection of all girls in your school.
4. The collection of five weakest countries in the world.
5. The collection of all real numbers.
6. The collection of all odd integers from 1 to 100.
7. The collection of all circles in a plane.
8. The collection of all equilateral triangles in a plane.
9. The collection of all hair on your body.
10. The collection of all the grains of sand in the entire world.

7.2 Notation : Ways of Describing a Set

Sets are usually denoted by the capital letters A, B, C, X, Y, etc. Their elements are denoted by the small letters a, b, c, etc.

Let A be the set of all one-digit natural numbers. Then 4 belongs to the set or is an element of the set. We express this in symbols as

$4 \in A$ and read it as ' 4 belongs to A '.

The symbol $\in$ stands for the words 'belongs to' or 'is an element of'. Evidently, 10 does not belong to the set. We use the notation

$10 \notin A$ and read it as ' 10 does not belong to A ' to express this fact.

In general, if an element a belongs to a set A , we write $a \in A$, and if it does not, then we write $a \notin A$.

We need some notation for describing sets. There are two ways of describing sets.

1. Tabular method : In this method we list all the members of the set separated by commas and enclose them in braces or curvilinear brackets. Thus, if A is the set of all one-digit odd integers from 1 to 10, then its elements are 1, 3, 5, 7, 9, and we write the set in the *tabular form* as

$$A = \{1, 3, 5, 7, 9\}$$

Again, sets of examples 2, 4, 5, 8 of Section 7.1 can be written in this form as

2. $\{\text{mar, wife, first son, second son, daughter}\}$
4. $\{a, b, c, j, \dots, x, y, z\}$, the dots standing for the unlisted letters.
5. $\{3, 4, j, 6, 7, 8, 9\}$
8. $\{S, C, H, O, L\}$

The tabular form of a set is also called the *Roster form* since all the elements are listed or what is the same thing, a roster of the elements is given. This method has the advantage that one can visualize the elements and thus visualize the set as a whole. However, the method is not useful when the number of members of the set is very large, or when the rule regarding membership of the set does not permit the listing of all the members.

We may remark here that the *order of writing the elements of a set is immaterial*. Thus the following sets are the same.

$$\{a, b, c\}, \{c, a, b\}, \{b, c, a\}$$

Set-builder method: When the elements of a set are described by a common property or a rule, we denote a typical member of the set by x, y , etc., follow it up by the statement of the rule and enclose the whole in curvilinear brackets as before. Thus, if A is the set of all integers, we write

$$A = \{x \mid x \text{ is an integer}\}$$

This is read as 'the set of all x such that x is an integer'. In this form, we may understand the first bracket ' $\{$ ' to mean 'the set of all', the standing line ' $\mid$ ' to mean 'such that', the final bracket ' $\}$ ' being added for the sake of symmetry. Sometimes, a colon ':' is also used in place of ' $\mid$ '.

The sets in examples 5, 6, 15 of Section 7.1 can be written in the set-builder form as

$$5. \{x \mid x \text{ is a natural number and } 2 < x < 10\}$$

$$6. \{n \mid n \text{ is an even natural number}\}$$

$$15. \{x \mid x \text{ is a real number}\}$$

Since the rule defining the set is written in this form of describing a set, it is also called the *Rule form*.

EXERCISE 7.2

1. Let $A = \{x \mid x \text{ is a natural number and is a multiple of } 7\}$
 - (a) List the elements of A which are smaller than 50.
 - (b) Insert the symbol $\&$ or $\in$ as appropriate in the blank spaces represented by dashes in the following :

- | | | |
|-------------|--------------|--------------|
| (i) $5-A$ | (ii) $13-A$ | (iii) $51-A$ |
| (iv) $21-A$ | (v) $84-A$ | (vi) $28-A$ |
| (vii) $0-A$ | (viii) $7-A$ | (ix) $35-A$ |

2. Write the following sets in the tabular form :

- (i) $A = \{x \mid x \text{ is an integer and } -5 < x \leq 4\}$.
- (ii) $B = \{x \mid x \text{ is a two-digit natural number the sum of whose digits is } 7\}$.
- (iii) $C = \{x \mid x = 0 \text{ or } x = 1\}$
- (iv) $D = \{x \mid x \text{ is an integer, } 3 \leq x < 15\}$
- (v) The set of all letters in the word 'INDIA'.

3. Write the following sets in the set-builder form :

- (i) $\{a, e, i, o, u\}$
- (ii) $\{-3, -2, -1, 0, 1, 2\}$
- (iii) $\{4, 8, 12, 16, 20, 24\}$
- * (iv) $\{1, 4, 9, 16, 25, 36, \dots\}$

7.3 Finite and Infinite Sets : the Empty Set

In many sets, the number of elements can be counted. For example, the set of letters of the English alphabet contains the 26 elements $a, b, c, \dots, x, y, z$. The sets of examples 2, 5 of Section 7.1 contain 5, 7 members respectively. The number of elements in examples 1, 3 and 11 cannot be stated immediately, but these elements can be counted and we can be sure that it is a definite number. All these are examples of *finite sets*. A finite set contains a definite number of elements.

A set which is not finite is called an infinite set. In such a set, the number of elements cannot be expressed by any

positive integer. Can you see that the sets in examples 6, 12-15 are infinite? In example 6, we can count the elements as first, second, third, etc., but the counting process never comes to an end. In others, even this cannot be done. Why? Can you recall the property that between any two distinct real numbers, there lie as many real numbers as we please?

Sometimes the rule determining the elements of a set may be such that no objects satisfy the condition laid down by the rule. For example, consider the set of all natural numbers which are less than 5 and also greater than 6. Evidently, there is no such natural number. So there are no elements in the set. We may very well say that there is no set satisfying the given conditions. However, it is found convenient to say that this is also a set but without elements. It is called **the empty set** or **the null set**. Whenever the rule for determining the elements of a set involves some contradiction in itself, we get the empty set. We denote the empty set by the symbol ϕ . The symbol ' $\{ \}$ ', without anything between the brackets, is also sometimes used to denote the empty set.

A set consisting of a single element is sometimes called a **singleton set**. Examples are

$\{\text{Sun}\}$, $\{\text{Earth}\}$, $\{5\}$, $\{6\}$, etc.

You can think of other examples.

EXERCISE 72

1. Which of the following are finite and which are infinite sets?
 - (i) The set of all the divisors of 64.
 - (ii) The set of multiples of 4.

- (iii) The set of all natural numbers which have 6 in the unit's place.
- (iv) The set of all real numbers greater than 6.
- (v) The set of integers between -50 and 50 .
- (vi) $\{x \mid x \text{ is the cube of a natural number and } x \leq 1000\}$.

2. Which of the following is the empty set ?

- (i) The set of Mathematics books in your school library.
- (ii) $\{x \mid x \text{ is an integer and } 2x = -3\}$
- (iii) $\{x \mid x \text{ is an even prime}\}$

7.4 Equality of Sets; Subsets

We now learn some relations between sets.

Equality of Sets : Two sets are said to be equal if they have the same elements.

Thus equality is interpreted in the sense of sameness. We use the symbol ' $=$ ' for equality as for numbers. We give some examples of equal sets.

Example 1 : $\{a, b, c\} = \{b, a, c\}$

Example 2 : If $A = \{x \mid x = n^2, n \text{ is a natural number}\}$, and $B = \{1, 4, 9, 16, 25, \dots\}$, where the dots stand for the following perfect square natural numbers, then $A = B$.

Can you observe that when two sets are equal, every element of one is an element of the other and vice versa ?

Subsets : In place of the relations 'greater than' and 'less than', we now have the relation of a set being a *subset of another set*. We consider some examples.

Example 3 : Consider the sets

$$A = \{a, b, c, d, e, f\}, B = \{a, b, c, d, e, f, g, h\}$$

We see that every element of A is also an element of B .

Example 4 : Consider the sets $A = \{1, 3, 5, 7, 9\}$ and $B = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Again, every element of A is an element of B .

In both these examples, we say that A is a subset of B . In these examples, B contains elements which are not in A . This is not necessary always. All that is necessary to make A a subset of B is that every element of A should be an element of B .

We thus say :

A set A is a subset of a set B if every element of A is an element of B . We write this in symbols as

$$A \subset B$$

and read it as ' A is a subset of B ' or ' A is contained in B '. The symbol $\subset$ is also called the *inclusion* symbol.

When A is a subset of B , we sometimes say that ' B is a superset of A '. We write it in symbols as

$$B \supset A \text{ and read it as 'B contains A'}$$

Remarks : 1. Observe that every set can be regarded as a subset of itself. Thus, in symbols

$$A \subset A \text{ is true for every } A.$$

2. The empty set contains no elements; so it can be regarded as a subset of every set. Thus, in symbols,

$$\phi \subset A \text{ is true for every } A.$$

3. We observe that we cannot write ' $x \subset A$ ' for ' $x \in A$ ' when x is an element of A .

4. Can you observe that the symbols $\subset$ and $\supset$ look like elongated and rounded $<$ and $>$? Does it give you an idea about their origin?

We now take a few more examples.

Example 5: Consider the sets $A = \{a, b, c, e, f\}$ and $B = \{c, d, e, f, g, h\}$. Is every element of A a member of B ? No. So A is not a subset of B . Can you check that B is also not a subset of A .

When a set A is not a subset of a set B , we write it in symbols as $A \not\subset B$ and read it as ' A is not contained in B ' or ' A is not a subset of B '. Here, $A \not\subset B$ and $B \not\subset A$.

Example 6: If $A \subset B$ and $B \subset A$, then $A = B$.

Solution: For, if every element of A is an element of B , and every element of B is an element of A , then A and B must have all their elements the same. Thus $A = B$.

The converse of this is also true. Can you see this?

7.4.1. Universal Set: It often happens, in discussions involving sets, that all the sets under consideration are subsets of a larger set. Then this larger set of which all the others are subsets is called *the universal set* or *the universe of discourse*, for the discussion at hand. For example, if we are making population studies in our country, our universal set is the set of all citizens in our country. Again, if we are discussing sets of points on a line, our universal set is the set of all the points on the line. If we are working with real numbers, our universal set is the set of all real numbers. In all such discussions, the universal set is given before hand either implicitly or explicitly.

Example 7: Write down all the subsets of $\{1, 2, 3\}$.

Solution: In forming the subsets, we may take one of its elements at a time or two at a time or all three. Thus we get the subsets

$\{1\}, \{2\}, \{3\}, \{2, 3\}, \{3, 1\}, \{1, 2\}, \{1, 2, 3\}$.

These are seven in number. Also we have agreed to re-

gard the empty set ϕ as a subset of every set. We add this to the seven already listed to get eight subsets.

EXERCISE 7.4

1. Let $X = \{-2, -1, 0, 1, 2\}$, $Y = \{0, -1, 2, -2, 1\}$,
 $A = \{1, 2, 3\}$, $B = \{-3, -1, 2\}$, $C = \{-1, 1, 0\}$
 $D = \{-1, 1, 2\}$, $E = \{-2, -1\}$.

(i) Which of the sets Y, A, B, C, D, E are subsets of A ?

(ii) Are the sets X and Y equal?

Write down all the subsets of the following sets.

2. $\{-1, 1\}$. 3. $\{0, 1, 2\}$. 4. $\{a, b, c, d\}$
 5. How many subsets has each of the sets in Questions 2-4?

6. Which of the following statements are true and which false?

(i) $a \in \{c, f, j\}$

(ii) $\{a\} \subset \{a, b, c\}$

(iii) $\{a\} \supset \{a\}$

(iv) $\{0, 1\} \in \{0, 1, 2\}$

(v) $\{x, y, z\} \subset \{x, y\}$.

7. Can we take the following as a definition of a subset?
 A is a subset of B if $x \in A$ implies $x \in B$.

8. Make correct statements by filling in the symbols $\subset$ or $\not\subset$ in the blank spaces indicated by dashes.

(i) $\{-1, 0, 1\}$ — $\{-1, 0, 2\}$

(ii) $\{-1, 0, 1\}$ — $\{0, -1, 1\}$

(iii) $\{x \mid x \text{ is an even natural number}\}$ — $\{x \mid x \text{ is an integer}\}$

(iv) $\{1, 3, 6, 7\}$ — $\{3, 5, 7, 9, 6, 11\}$

7.5 Set Operations

We now describe two operations on sets which take the place of addition and multiplication for real numbers. Since these differ very much from ordinary addition and multiplication, we use new names and symbols for these.

The first is the operation of taking '*the union of two sets*'. As you know the word union means joining together of two things. So how do we interpret this for two sets. Let us take two examples.

Example 8 : Consider the sets $A = \{1, 2, 3\}$ and $B = \{4, 5, 6, 7\}$. If we put all their elements together in one set, we get the set $C = \{1, 2, 3, 4, 5, 6, 7\}$. Would you agree that C is the set obtained by the joining together or union of A and B ?

Example 9 : Consider now the sets $A = \{x \mid x \text{ is real and } 1 < x < 3\}$ and $B = \{x \mid x \text{ is real and } 2 < x < 4\}$. If we put together all the elements of A and B in a new set, we see that we get all the real numbers between 1 and 4 with the numbers $2 < x < 3$ each repeated once. Since repetitions are not allowed in sets, we get the set

$$C = \{x \mid x \text{ is real and } 1 < x < 4\}$$

In both the above examples, the set C is called the *union set* of the sets A and B . We thus have :

Union of Sets : The union of two sets A and B is the set C which consists of all those elements which are in A or in B or in both. We write

$C = A \cup B$, and read it as *A union B*. In symbols,

$$C = A \cup B = \{x \mid x \in A \text{ or } x \in B \text{ or } x \in \text{both}\}.$$

The second operation on sets is that of '*taking*' the *intersection of two sets*'. The word intersection means where two things meet, such as the intersection of two roads or two

straight lines or two circles. In other words, we have to find what is common in the two sets. Let us consider two examples.

Example 10: Consider the sets

$$A = \{1, 2, 3, 4, 5, 6\}, \quad B = \{4, 5, 6, 7, 8\}.$$

What is common between the two sets? Evidently, the elements 4, 5, 6. They form the set $C = \{4, 5, 6\}$. So C is the set of common elements of A and B .

Example 11: Consider the sets

$$A = \{a, b, c, d\}, \quad B = \{f, g, h, j, k\}$$

Now what is common between the two sets?

Evidently, nothing. So what is the set formed by their common elements? The set $C = \phi$.

In both the above examples, the set C is called the *intersection set* of the sets A and B . We thus have

Intersection of Sets: The intersection of two sets A and B is the set C which consists of all those elements which are in A as well as in B . If there is no such element, then the intersection is the empty set ϕ . We write

$$C = A \cap B \text{ and read it as } A \text{ intersection } B.$$

In symbols,

$$C = A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

We now give some more examples.

Example 12: Let $A = \{x \mid x \text{ is an integer and } 3 < x \leq 5\}$ and $B = \{x \mid x \text{ is an integer and } 4 \leq x < 8\}$, then $A \cup B = \{x \mid x \text{ is an integer and } 3 < x < 8\} = \{4, 5, 6, 7\}$ and $A \cap B = \{x \mid x \text{ is an integer and } 4 \leq x \leq 5\} = \{4, 5\}$.

Example 13: If $A = \{1, 2, 3, 4, 5\}$ and $B = \{6, 7, 8\}$ then $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A \cap B = \phi$.

When two sets have no element in common, that is their intersection is the empty set, then the sets are said to be

disjoint. Thus the sets in Example 13 above are disjoint.

Example 14 : If A is any set, then both its union with itself and its intersection with itself are evidently equal to A itself. Thus,

$$A \cup A = A \text{ and } A \cap A = A \text{ for every set } A.$$

EXERCISE 7.5

- Let $A = \{-1, 0, 2, 3\}$, $B = \{3, 4, 5, -3\}$, and $C = \{2, 4, 6, 8\}$. Find
 - $A \cup B$
 - $B \cup C$
 - $C \cup A$
 - $A \cap B$
 - $B \cap C$
 - $C \cap A$
 - $(A \cup B) \cup C$
 - $A \cup (B \cup C)$
 - $(A \cap B) \cap C$
 - $A \cap (B \cap C)$
- Find the union and intersection of $\{x \mid x \text{ is a natural number}\}$ and $\{x \mid x \text{ is an integer}\}$.
- Find the union and intersection of the sets $\{x \mid x \text{ is an odd natural number}\}$ and $\{x \mid x \text{ is an even natural number}\}$.
- Which of the following pairs of sets are disjoint and which not?
 - $\{1, 2, 3, 4\}$ and $\{1, 3, 4, 5, 6\}$
 - $\{a, e, i, o, u\}$ and $\{c, d, e, f\}$
 - $\{x \mid x \text{ is an even integer}\}$ and $\{x \mid x \text{ is an odd integer}\}$
 - $\{x \mid x \text{ is a prime}\}$ and $\{x \mid x \text{ is even and } x > 2\}$
- Let A, B be any two sets. Can we say that $A \cup B$ and $A \cap B$ are disjoint always? Support your answer by examples.
- In each of the following pairs of sets, remove the minimum number of elements from one of the

two sets so that the resulting pair is disjoint.

- (i) $\{5, 9, 13, 23, 27\}$ and $\{3, 5, 7, 9, 37, 41\}$
- (ii) $\{-1, 0, 1, 2, 3\}$ and $\{3, -3, -2, 0\}$
- (iii) $\{a, e, i, o, u\}$ and $\{a, b, c, d, e, f\}$

7.6 Properties of Set Operations

Although the operations of union and intersection defined on sets are not similar to the operations of addition and multiplication defined on numbers, they have many of the properties of addition and multiplication.

Union and intersection of sets are also commutative, associative and distributive. For example, it is immaterial whether we form the union of two sets A and B or of B and A. Similarly, for intersection of A and B. Can you verify these for any two sets of your choice?

We will study these properties in detail in higher classes.

7.7 Venn Diagrams

We have studied the concept of a set, some relations between sets, operations on sets and the properties of these operations. All these become clearer if we use *Venn Diagrams** to represent sets pictorially. In these diagrams, the universal set, of which all the sets under consideration are the subsets, is represented by the interior of a rectangle, and any given set is represented by the interior of a circle drawn inside the rectangle. We could also use triangles, squares and the like

* These diagrams are named after the English mathematician John (1834-1923).

in place of circles. Fig. 7.1 (i), (ii), (iii) gives the representation of two sets A and B in the three cases when the sets are not disjoint, disjoint and when $A \subset B$. Notice that the circle representing A is drawn inside the circle representing B in the third case.

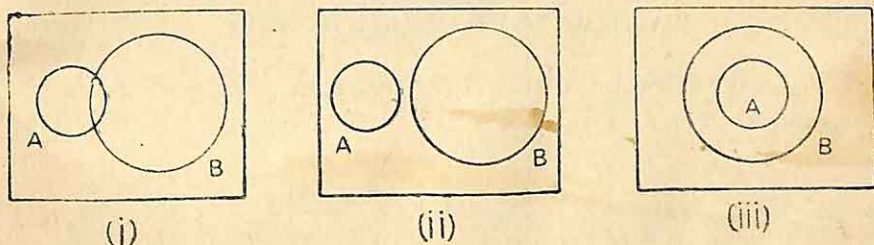


Fig. 7.1

The shaded areas in Fig. 7.2 (i), (ii), (iii) show $A \cup B$ in the same three cases.

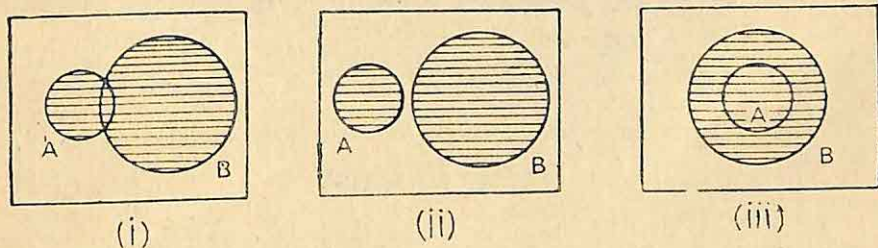
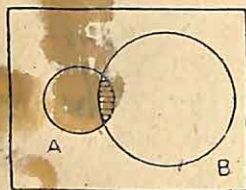
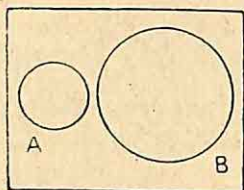


Fig. 7.2 Venn diagram for $A \cup B$

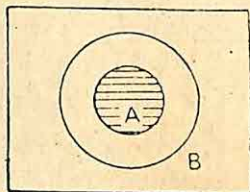
Fig. 7.3 gives the pictorial representation of $A \cap B$ in the three cases.



(i)



(ii)



(iii)

Fig. 7.3 Venn diagram for $A \cap B$

Notice that when A and B are disjoint, there is no shaded area in Fig. 7.3 (ii)

EXERCISE 7.6

If A and B are sets, draw Venn diagrams for the following :

1. $A \cap B$ when $B \subset A$
2. $A \cup B$ when $B \subset A$
3. Verify the following with the help of Venn diagrams
 - (i) $A \cup B = B \cup A$
 - (ii) $A \cap B = B \cap A$

Key Concepts

Set

Finite and infinite sets

Null set

Subsets

Universal set

Union of sets

Intersection of sets

Disjoint sets

Venn diagrams

Miscellaneous Exercise II

(On Units V and VII)

Draw the graph of the following equations :

1. $3x = -5$

2. $2y = 3$

3. $y - x = 0$

4. $x - y + 5 = 0$

5. $\frac{x}{2} + \frac{y}{3} = 1$

Draw the graph of the following inequations :

6. $2x \leq 1$

7. $x \geq -2$

8. $y > -3$

9. $3y + 9 \leq 0$

Solve the following simultaneous equations graphically :

10. $4x - 3y + 3 = 0$

11. $2x + 3y + 7 = 0$

$2x + y - 11 = 0$

$3x - y + 5 = 0$

Solve the following simultaneous equations :

12. $4x - y = 0$

13. $2x - 3y + 5 = 0$

$9x - y = 40$

$3x + 2y - 6 = 0$

14. $2x - 3y - 1 = 0$

15. $\frac{x}{7} - \frac{y}{6} + \frac{4}{21} = 0$

$3x + 4y - 27 = 0$

$\frac{x}{13} + \frac{y}{5} - \frac{6}{65} = 0$

16. The sum of the ages of the father and his son is 48 years. After 21 years father will be twice as old as his son. Find their present ages.

17. Which of the following are sets ? For those that are not, explain why ?

- (i) The collection of your school teachers whose names begin with the letter S.
- (ii) The collection of wise girls.
- (iii) The collection of books written by Tulsidas.

18. Write the following sets in the tabular form :

- (i) The set of vowels in the English alphabet.
- (ii) $A = \{x \mid x \text{ is prime and } x < 20\}$
- (iii) $B = \{x \mid x \text{ is a day of the week}\}$

19. Write the following sets in the set-builder form :

- (i) $\{6, 12, 18, 24, 30\}$
- (ii) $\{2, 4, 6, 8, 10\}$

20. Write down all the subsets of the set $\{a, b, c\}$.

21. Let $A = \{x \mid x \text{ is an even natural number and } x \leq 10\}$.
 $B = \{x \mid x \text{ is an odd natural number and } x \leq 10\}$.
Find $A \cup B$ and $A \cap B$.

22. Which of the following pairs of sets are disjoint and which not ?

- (i) $\{1, 4, 9\}$ and $\{2, 4, 6, 7\}$
- (ii) $\{2\}$ and $\{3\}$
- (iii) $\{a, b, c, f\}$ and $\{f, g, h\}$

23. If A, B, C are sets then draw Venn diagrams for the following :

- (i) $(B \cap C)$ when $B \subset C$
- (ii) A and C are disjoint sets ; A and C are subsets of B.

EXERCISE 1.1

1. 100003; 2. 100; 3. 50 cm, 16; 4. 18, 50, 19; 5. 2, 3,
5, 7, 11, 13, 17, 19, 23, 29; 7. $\frac{5}{3}$, $\frac{12}{13}$, $\frac{14}{5}$; 8. $\frac{211}{168}$;
9. $-\frac{3}{5}$, $\frac{8}{11}$, $\frac{11}{15}$, $\frac{7}{9}$; 10. $1.\overline{714285}$, $-1.\overline{6}$, $-0.91\overline{6}$;
11. $\frac{5}{8}$; 13. $\frac{67}{33}$; 15. -1715, 987003; 16. 202, 204;
17. odd; 18. even; 21. (i) True, (ii) False.

EXERCISE 1.2

2. 1.7320; 4. (i) Rational, (ii) Irrational, (iii) Irrational,
(iv) Irrational, (v) Irrational; 5. 3.35663; 6. 2.4393; 7. 0.707;
8. $2.4\overline{29}$; 9. $5-2\sqrt{6}$; 10. $\sqrt{5}+\sqrt{3}$.

EXERCISE 2.1

1. (i) $\sqrt{2}$, 3, (ii) $\sqrt{2}$, 2, (iii) 2, 1, (iv) a^2 , -3, (v) a^{-2} , 2, (vi) a , -6,
(vii) $\sqrt{2}$, 0, (viii) 1, 1; 2. (i) $(\sqrt{2})^4$, (ii) $(\sqrt{3})^2$, (iii) $(4)^2$, (iv) $\left(\frac{3}{4}\right)^1$,

- (v) $(a^{-2})^3$; 4. (i) 1, (ii) 3, (iii) $\frac{1}{4}$, (iv) $9\sqrt{3}$, (v) $\sqrt{2}$.

EXERCISE 2.2

1. (i) 32, (ii) 4, (iii) $\frac{1}{9}$, (iv) $\frac{1}{125}$, (v) 1024, (vi) $2\sqrt{3}$,
(vii) $\frac{\sqrt{2}}{4}$, (viii) 8; 2. (i) 36, (ii) $\frac{19}{2}$.

EXERCISE 2.3

1. (i) $\frac{3a^2}{b^4}$, (ii) $\frac{1}{a^{16}}$, (iii) $\frac{a^{12}}{b^2}$, (iv) $a^{\frac{2}{3}}$; 2. (iv), (vii), (xii);
3. (i) 4, (ii) 4, (iii) 2.

EXERCISE 2.4

1. (iv), (vi); 2. (i) $(\sqrt{2})^{-12}$, (ii) $(\sqrt{2})^{12}$, (iii) $(\sqrt{3})^{-9}$,
(iv) $(\sqrt{5})^{\frac{12}{5}}$; 3. $\frac{1}{2}$; 4. $\frac{16}{3}$.

EXERCISE 2.5

1. $32-\sqrt{2}$; 2. 3; 3. $\sqrt{5}$; 4. -11; 5. $7\sqrt{7}$;

6. $\frac{9\sqrt{70}}{2}$; 7. $\frac{9}{2}$; 8. $\frac{5\sqrt{2}}{6}$; 9. $a^2\sqrt{b}$; 10. a ; 11. $\frac{b}{a}$;
 12. $\sqrt{ab}$; 13. $(12)^{\frac{3}{4}} a^{\frac{3}{4}} b^{\frac{3}{4}}$; 14. $3a+9a^3$; 15. $\sqrt{a}+\sqrt{2}$;
 16. $a-b$.

EXERCISE 3.1

1. (i) 9, (ii) 25, (iii) 4;
 2. $\frac{2}{11}$, $-\frac{8}{9}x$, $\frac{101}{10}x^5$, $\frac{99}{100}x^7$;
 3. (i) $-\frac{17}{14}y^3 - \frac{39}{28}y^2 + \frac{97}{56}y$;
 (ii) $\frac{100}{9}z^5 - \frac{790}{99}z^3 + z^2 + \frac{5}{6}z + 6$;
 4. $-\frac{2}{7}y^5 + y^3 - \frac{12}{5}y^2 - \frac{83}{7}y$;
 5. $\frac{7}{8}y, \frac{9}{4}yz, -\frac{15}{11}tz$;
 6. $-3.6x^4 + 11.6x^3 + 11.4x^2 + 8$;
 8. It can be of same or smaller degree;
 9. $-4x^2 - 2x + 1$; 10. $-2x^2 + x - 3$.

EXERCISE 3.2

1. $-\frac{1}{3}x^4, \sqrt{2}x^5, \frac{6}{1.5}x^7, \frac{3}{17}x^{11}$; 2. $1.2a, -\frac{1}{\sqrt{3}}b, -\frac{\sqrt{2}}{\sqrt{5}}cd$;
 3. (i) $\sqrt{5} + \frac{6}{5}x + \left(\frac{1}{3} - \frac{2}{\sqrt{7}}\right)x^2 - \frac{13}{15}x^3$.

- (ii) $8 - \frac{1}{7}y + \frac{1}{\sqrt{7}}y^2 + \sqrt{11}y^4 - y^{11}$;
4. $\left(\frac{1}{\sqrt{3}} - 1\right)y^{99} + \left(\frac{2}{\sqrt{3}} + \frac{1}{3}\right)y^{98} - 6 - \sqrt{17}$;
5. $2 + \frac{1}{6}x + \frac{7\sqrt{2}}{2}x^2$;
6. $\left(\pi - \sqrt{\frac{6}{5}}\right)x^2 + \left(\sqrt{\frac{3}{5}} - \frac{\pi}{3}\right)x - \sqrt{3}$.

EXERCISE 3.3

1. (i) x^{11} , (ii) x^8 , (iii) x^3 , (iv) x^{20} ; 2. (i) x^5 , (ii) x^6 , (iii) x^9 , (iv) x^5 .

EXERCISE 3.4

1. $\frac{3}{8}x^8$; 2. $\frac{\sqrt{15}}{42}x^8$; 3. $-\frac{1}{7\sqrt{6}}x^7$; 4. $\frac{9}{11}x^{15}$;
5. $\frac{3}{2}x^{13}$; 6. $\left(\frac{2+4\sqrt{2}}{\sqrt{3}}\right)x^8$; 7. $4x^6$; 8. $\sqrt{3}x^2$;
9. $-\frac{64}{77}x^{14}$; 10. $\frac{81}{5\sqrt{5}}x^0$.

EXERCISE 3.5

1. $\frac{\sqrt{2}}{2}x^3 + 4\sqrt{2}x^2$; 2. $-\frac{3}{2}x^3 - \frac{\sqrt{3}}{4}x^2$; 3. $\frac{1}{6}x^8 + \frac{\sqrt{2}}{33}x^5$;
4. $-\frac{15}{11}x^4 - \frac{35}{33}x$.

EXERCISE 3.6

1. $x^2 + (a+1)x + a$; 2. $\frac{\sqrt{2}}{6}x^3 + \left(\frac{2+3\sqrt{2}}{6}\right)x^2 + x$;
 3. $x^3 + 2.5x^2 + 1.7x - 2$; 4. $\left(\frac{11-\sqrt{11}}{11}\right)x^2 - \left(\frac{22+3\sqrt{5}}{3}\right)x - \frac{2\sqrt{5}}{3}$;
 5. $\frac{23}{16}x^2 - \frac{1}{8}x + \frac{37}{162}$; 6. $\frac{1}{3}z^4 + \frac{5}{6}z^3 + \frac{29}{54}z^2 - \frac{7}{18}z + \frac{1}{9}$;
 7. $1 + \left(\sqrt{2} + \frac{1}{\sqrt{6}}\right)z - \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}\right)z^2 - z^3$.

EXERCISE 3.7

1. $x+6$; 2. No; 3. (i) $y^2 - y + 1$, (ii) $y+1$;
 4. $5x^2 - 7x + \frac{2}{3}$, $\frac{43}{3}x - \frac{53}{3}$; 5. $2x^2 + 1$.

EXERCISE 3.8

1. (i) Rational expression, (ii) Polynomial, (iii) Rational expression,
 (iv) Polynomial, (v) Rational expression.
 3. $-10x^3 + \frac{24}{7}x^2 - 7x + \frac{3}{2}$.

EXERCISE 3.9

1. (i) $\frac{2x^2+2x}{2x-1}$ (ii) $\frac{-x^4+2x^3+4x^2+4x-1}{x^3+2x^2+x+2}$,

$$(iii) \frac{2-2\sqrt{6}x^2}{1-(\sqrt{2}+\sqrt{3})x+\sqrt{6}x^2}$$

$$(ii) \frac{y^2-2y+2}{1-y}$$

$$2. \frac{8y^3+2y+2}{1-y-4y^3}; 3. (i) \frac{2x^3-2x^2+x}{x-1};$$

EXERCISE 3.10

$$1. \frac{4x^3+4x^2+9x+2}{4x^3+4x^2+3x+1}; 2. \frac{3+13x+10x^2-3x^3}{2+3x};$$

$$3. \frac{128x^3+352x^2+102x-2}{16x^2+42x+5}.$$

EXERCISE 3.11

$$1. (i) \frac{10x^2+x-3}{5x^2+4x-1}, (ii) \frac{2x^2+5x+2}{3x^2+3}, (iii) \frac{5y^2+15y}{72y^2+14y-1},$$

$$(iv) \frac{8x+7x^2+6}{x+1}; 2. \frac{29+39x-20x^2-15x^3}{x^2+3x-4};$$

$$3. \frac{60+35y+55y^2-7y^3}{5+6y}.$$

EXERCISE 3.12

$$1. (i) \frac{3x+0.1}{0.5x+0.7}, (ii) \frac{7x^2-2x+0.3}{8x^2+7x+0.1}, (iii) \frac{3y+0.8}{20y-8y^2+5};$$

$$2. \frac{x^4+40x^3+421x^2+402x+101}{x^3+21x^2+30x+10}; 4. (i) \frac{1}{2x+3}, (ii) x+1.$$

EXERCISE 3.13

1. $\frac{2x^2+3x+1}{x^2-2x+1}$; 2. $\frac{3y^2+2y-5}{5+3y-2y^2}$;
 3. (i) $\frac{4x^4+6x^3-2x^2+4x}{x^4-2x^2+1}$, (ii) $3y^4+2y^3+5y^2-\frac{9}{4}y$;
 4. (i) $\frac{x^2+x+1}{x^2+x}$, (ii) $\frac{x^2-x-1}{x^2+x}$, (iii) $\frac{1}{x+1}$, (iv) $\frac{x^2}{x+1}$.

EXERCISE 4.1

1. x^2+2x+1 ; 2. $4x^2+12x+9$; 3. y^2-2y+1 ; 4. x^2 ;
 5. 9801; 6. $4x^2+12xz+9z^2$; 7. $4x^2+4x+1$; 8. $9x^2-6x+1$;
 9. $4z^2-20z+25$; 10. $x^2+2\sqrt{2}x+2$; 11. $x^2+\sqrt{10}x+\frac{5}{2}$;
 12. $x^2+\frac{9\sqrt{2}}{2}x+9$; 13. $\sqrt{2}d^3+4\sqrt{2}d^2$; 14. $2-\frac{1}{2}x^2$.

EXERCISE 4.2

1. $a^3+3a^2b+3ab^2+b^3$; 2. $a^3+6a^2b+12ab^2+8b^3$;
 3. $x^3+6x^2+12x+8$; 4. $x^3+6x^2y+12xy^2+8y^3$;
 5. $216x^3+756x^2y+882xy^2+343y^3$; 6. $x^3+\sqrt{3}x^2+x+\frac{1}{3}$;
 7. $27x^3+27x^2+9x+1$; 8. 1061208; 9. 8615125; 10. 1003003001;
 11. 1157.625.

EXERCISE 4.3

1. $a^3 - 3a^2b + 3ab^2 - b^3$; 2. $8a^3 - 60a^2b + 150ab^2 - 125b^3$;
3. $512x^3 - 192x^2 + 24x - 1$;
4. $1 - \frac{21}{10}x + \frac{147}{100}x^2 - \frac{343}{1000}x^3$; 5. $8x^3 - 36x^2z + 54xz^2 - 27z^3$;
6. 912673; 7. 997002999; 8. 970.299.

EXERCISE 4.4

1. $a^3 + 8$; 2. $a^2 + 1$; 3. $0.343x^3 + 0.729y^3$; 4. $8x^3 + 343$;
5. $\frac{x^3}{8} + \frac{27}{64}y^3$.

EXERCISE 4.5

1. $1 - x^3$; 2. $125x^3 - 27y^3$; 3. $8a^3 - 1$; 4. $1 - 8a^3$;
5. $27x^3 - \frac{1}{343}y^3$.

EXERCISE 4.6

1. $\sqrt{2}x(1 + \sqrt{2}x)$; 2. $7y(1 + 3y - 7y^2)$; 3. $(x - \sqrt{2})(x + \sqrt{2})$;
4. $(\sqrt{7}y - \sqrt{3}z)(\sqrt{7}y + \sqrt{3}z)$; 5. $(y + 2)^2$; 6. $(0.1x + y)^2$;
7. $(x - \sqrt{3})^2$; 8. $(\sqrt{2}x - y)^2$.

EXERCISE 4.7

1. $(x+1)^2$; 2. $(x-5)^2$; 3. $(x+1)(x+5)$; 4. $(y-3)(y+1)$;
 5. $(z-3)(z-2)$; 6. $\left(y + \frac{1}{3}\right)\left(y + \frac{1}{4}\right)$; 7. $(x+5)(x-4)$;
 8. $(x+a)(x+2a)$; 9. $(x+3k)(x-k)$; 10. $(x-3p)(x+2p)$.

EXERCISE 4.8

1. $(a-1)(a^2+a+1)$; 2. $a(a+1)(a^2-a+1)$;
 3. $(2x+7y)(4x^2-14xy+49y^2)$; 4. $\left(\frac{1}{6}a+2b\right)\left(\frac{1}{36}a^2-\frac{1}{3}ab+4b^2\right)$;
 5. $\left(\frac{1}{\sqrt{5}}x-\sqrt{2}y\right)\left(\frac{1}{5}x^2+\sqrt{\frac{2}{5}}xy+2y^2\right)$;
 6. $(0.1x-0.5y)(0.01x^2+0.05xy+0.25y^2)$.

MISCELLANEOUS EXERCISE I

4. 2.8284; 7. Rational number; 9. (i) 55.9, (ii) 5, (iii) 0.037, (iv) 127;
 10. (i) $\frac{a^2b^2}{5^8}$, (ii) $\frac{2^5a^{25}}{b^5}$; 11. (ii), (iv); 12. (i) 5, (ii) -2, (iii) 1,
 (iv) $\frac{4+\sqrt{3}}{12}$; 13. (i) $(\sqrt{6})^{-2}$, (ii) $5^{\frac{7}{2}}$; 14. -4; 15. $-\frac{9}{2}$
 16. (i) 7, (ii) $2-\sqrt{3}$; 17. (i) $a^{\frac{5}{2}}b$, (ii) $2a^{\frac{7}{2}}b^2$;
 18. $4\sqrt{3}x^2+14x+5\sqrt{3}$; 19. $x^2+4x-5\sqrt{3}$; 20. $10x^2+\frac{13\sqrt{3}}{3}x-1$
 21. $1+4\sqrt{2}x-2x^2-\sqrt{2}x^3$; 22. $\sqrt{14}x+(\sqrt{7}-\sqrt{14})x^2-\sqrt{7}x^3$;
 23. Quotient is x^2+x , Remainder is $-2x-5$

24. Quotient is $3x^3 - 5x^2 + 2x + 4$, Remainder is $-6x - 6$;
 25. $\frac{2a}{a^2 - b^2}$; 26. $\frac{3x^2 + 6x + 2}{x^3 + 3x^2 + 2x}$; 27. $\frac{(a+b+c)x}{abc}$;
 28. $\frac{14x}{49x^2 - 5}$; 29. $\frac{1 + 15x + 6x^2 - x^3}{3x^3 - 9x}$; 30 (i) $3\sqrt{x}$, (ii) $\frac{4}{\sqrt{x}} - \sqrt{x}$;
 31. $(x-4)$ metres, 32. $\frac{x^2 - 2x - 2}{x^3 + x^2 - 2x}$; 33. $27x^3 + 125y^3$;
 34. $2\sqrt{2} x^3 - y^3$; 35. $(1+2a)(1-2a+4a^2)$;
 36. $(5p-1)(25p^2+5p+1)$; 37. $(4k+6t)(16k^2-24kt+36t^2)$;
 38. $(3r+d)(9r^2-3rd+d^2)$; 39. $3t^2-3t+1$; 40. 2.

EXERCISE 5.1

1. A(2,2), B(5,-2), C(3,-4), D(-3,-3), E(-5,-4), F(-4,3), G(-2,4); 2. No; 3. No.

EXERCISE 5.2

2. -2, 3; 3. $-\frac{3}{2}$, $-\frac{11}{2}$

EXERCISE 5.4

1. $x=3, y=3$; 2. $x=4, y=-6$; 3. No solution;
 4. $x=6, y=0$; 5. $x=4, y=-1$; 6. $x=2, y=-4$;
 7. No solution; 8. $x=4, y=1$.

EXERCISE 5.5

1. $x = \frac{1}{2}, y = \frac{3}{2}$; 2. $x=2, y=-3$; 3. $x = \frac{1}{3}, y = \frac{1}{2}$;
4. $x=16, y=1$; 5. $x=4, y=-1$; 6. $x=-2, y=-5$;
7. $x = \frac{22}{7}, y = \frac{1}{7}$.

EXERCISE 5.6

1. $x=45, y=15$; 2. $x=-\frac{5}{3}, y = \frac{17}{2}$;
3. $x = \frac{9}{10}, y = \frac{8}{5}$; 4. $x=1, y=1$; 5. $x=1, y=2$;
6. $x=2, y=6$; 7. $x = \frac{81}{19}, y = \frac{22}{19}$.

EXERCISE 6.1

1. 361; 2. 529; 3. 1331; 4. 4913; 5. 13824;
6. 2.645; 7. 4.242; 8. 4.582; 9. 2.289; 10. 2.466.

EXERCISE 6.2

1. Rs 2; 2. Rs 4 and paise 5; 3. Rs 20;
4. Rs 22 and paise 50; 5. Rs 3 and paise 21;
6. Rs 3.

EXERCISE 7.1

1, 3, 5, 6, 7, 8, 9, 10

EXERCISE 7.2

- (a) 7, 14, 21, 28, 35, 42, 49, (b) (i) $\notin$, (ii) $\notin$, (iii) $\notin$,
(iv) $\in$, (v) $\notin$, (vi) $\in$, (vii) $\notin$, (viii) $\in$, (ix) $\in$;
- (i) $\{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$, (ii) $\{16, 25, 34, 43, 52, 61, 70\}$,
(iii) $\{0, 1\}$, (iv) $\{3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$, (v) $\{I, N, D, A\}$;
- (i) $\{x \mid x \text{ is a vowel in the English alphabet}\}$,
(ii) $\{x \mid x \text{ is an integer and } -3 \leq x \leq 2\}$,
(iii) $\{x \mid x \text{ is a multiple of 4 and } 4 \leq x \leq 24\}$,
(iv) $\{x \mid x = n^2, n \text{ is a natural number}\}$.

EXERCISE 7.3

- (i) Finite, (ii) Infinite, (iii) Infinite, (iv) Infinite, (v) Finite
(vi) Finite ; 2. (ii).

EXERCISE 7.4

- (i) A, (ii) Yes ; 2. $\phi, \{-1\}, \{1\}, \{-1, 1\}$;
- $\phi, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{1, 2\}, \{0, 2\}, \{0, 1, 2\}$;
- $\phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\},$
 $\{a, b, d\}, \{b, c, d\}, \{c, d, a\}, \{a, b, c, d\}$; 5. 4, 8, 16 ; 6. (i) False,
(ii) True, (iii) True, (iv) False, (v) False ; 7. Yes ; 8. (i) $\nsubseteq$
(ii) $\subset$, (iii) $\subset$, (iv) $\nsubseteq$.

EXERCISE 7.5

1. (i) $\{-1, 0, 2, 3, 4, 5, -3\}$, (ii) $\{2, 3, 4, 5, -3, 6, 8\}$,
 (iii) $\{-1, 0, 2, 3, 4, 6, 8\}$, (iv) $\{3\}$, (v) $\{4\}$, (vi) $\{2\}$,
 (vii) $\{-1, 0, 2, 3, 4, 5, -3, 6, 8\}$, (viii) $\{-1, 0, 2, 3, 4, 5, -3, 6, 8\}$,
 (ix) ϕ , (x) ϕ ; 2. $\{x \mid x \text{ is an integer}\}$, $\{x \mid x \text{ is a natural number}\}$;
 3. $\{x \mid x \text{ is a natural number}\}$, ϕ ; 4. (i) Not disjoint, (ii) Not disjoint
 (iii) Disjoint, (iv) Disjoint; 5. No.

MISCELLANEOUS EXERCISE II

10. $x=3, y=5$; 11. $x=-2, y=-1$; 12. $x=8, y=32$;
 13. $x = \frac{8}{13}$, $y = \frac{27}{13}$; 14. $x=5, y=3$;
 15. $x = -\frac{62}{113}$, $y = \frac{76}{113}$; 16. 39, 9; 17. (i), (iii);
 18. (i) $\{a, e, i, o, u\}$, (ii) $\{2, 3, 5, 7, 11, 13, 17, 19\}$,
 (iii) $\{\text{Sunday, Monday, Tuesday, Wednesday, Thursday, Friday, Saturday}\}$;
 19. (i) $\{x \mid x \text{ is a multiple of 6 and } 6 \leq x \leq 30\}$, (ii) $\{x \mid x \text{ is an even natural number and } 2 \leq x \leq 10\}$;
 20. ϕ , $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$, $\{a, b, c\}$;
 21. $\{x \mid x \text{ is a natural number and } x \leq 10\}$, ϕ ; 22. (i) Not disjoint,
 (ii) Disjoint, (iii) Not disjoint.

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